SKINNER-RUSK UNIFIED FORMALISM FOR OPTIMAL CONTROL SYSTEMS AND APPLICATIONS

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May 15, 2007

Abstract

A geometric approach to time-dependent optimal control problems is proposed. This formulation is based on the Skinner and Rusk formalism for Lagrangian and Hamiltonian systems. The corresponding unified formalism developed for optimal control systems allows us to formulate geometrically the necessary conditions given by Pontryagin's Maximum Principle, providing that the differentiability with respect to controls is assumed and the space of controls is open. Furthermore, our method is also valid for implicit optimal control systems and, in particular, for the so-called descriptor systems (optimal control problems including both differential and algebraic equations).

Key words: Lagrangian and Hamiltonian formalisms; jet bundles, implicit optimal control systems, descriptor systems.

AMS s. c. (2000): 70G45, 49J15, 34A26, 49K15, 70H03, 70H05

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1 Introduction

In 1983 Skinner and Rusk introduced a representation of the dynamics of an autonomous mechanical system which combines the Lagrangian and Hamiltonian features [23]. Briefly, in this formulation, one starts with a differentiable manifold Q as the configuration space, and the Whitney sum $TQ \oplus T^*Q$ as the evolution space (with canonical projections $\rho_1 : TQ \oplus T^*Q \longrightarrow TQ$ and $\rho_2 : TQ \oplus T^*Q \longrightarrow T^*Q$). Define on $TQ \oplus T^*Q$ the presymplectic 2-form $\Omega = \rho_2^* \omega_Q$, where ω_Q is the canonical symplectic form on T^*Q, and observe that the rank of this presymplectic form is everywhere equal to $2n$. If the dynamical system under consideration admits a Lagrangian description, with Lagrangian $L \in C^\infty(TQ)$, then we obtain a (presymplectic)-Hamiltonian representation on $TQ \oplus T^*Q$ given by the presymplectic 2-form Ω and the Hamiltonian function $H = \langle \rho_1, \rho_2 \rangle - \rho_1^* L$, where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between vectors and covectors on Q. In this Hamiltonian system the dynamics is given by vector fields X, which are solutions to the Hamiltonian equation $i(X)\Omega = dH$. If L is regular then there exists a unique vector field X solution to the previous equation, which is tangent to the graph of the Legendre map $\mathcal{F}L : TQ \longrightarrow T^*Q$. In the singular case, it is necessary to develop a constraint algorithm in order to find a submanifold (in general only a subset) where there exists a well-defined dynamical vector field.

The idea of this formulation was to obtain a common framework for both regular and singular dynamics, obtaining simultaneously the Hamiltonian and Lagrangian formulations of the dynamics. Over the years, however, Skinner and Rusk's framework was extended in many directions. For instance, Cantrijn *et al* [7] extended this formalism for explicit time-dependent systems using a jet bundle language; Cortés *et al* [6] use the Skinner and Rusk formalism to consider vakonomic mechanics and the comparison between the solutions of vakonomic and nonholonomic mechanics. In [9, 13, 20] the authors developed the Skinner-Rusk model for classical field theories.

Furthermore, the Skinner-Rusk formalism seems to be a natural geometric setting for Pontryagin maximum principle. In this paper, whose roots are in the developments made in [7, 9, 13], we use a variation of the Skinner-Rusk formalism to study time-dependent optimal control problems. The wide range of application of our techniques enables geometrically implicit optimal control systems to be tackled, that is, systems where the control equations are implicit. In fact, systems of differential-algebraic equations appear frequently in control theory. Usually, in the literature it is assumed that it is possible to rewrite the problem as an explicit system of differential equations, perhaps using the algebraic conditions to eliminate some variables (for instance, in the case of holonomic constraints). However, in general, a control system is described as a system of equations of the type $F(t, x, \dot{x}, u) = 0$, where the x's denote the state variables and the u's the control variables, and there are some interesting cases where the system is not described by the traditional equations $\dot{x} = G(t, x, u)$. As examples, we consider the case of optimal control of Lagrangian mechanical systems (see [1, 3]) and also optimal control for descriptor systems [17].

The organization of the paper is as follows: Section 2 is devoted to giving an alternative approach of the Skinner-Rusk formalism for time dependent mechanical systems, carefully studying the dynamical equations of motion and the submanifolds where they are consistently defined. In Section 3 we develop the unified formalism for explicit time-dependent optimal control problems, and in Section 4 for implicit optimal control systems. Section 5 is devoted to examples and applications: first we study the optimal control of Lagrangian systems with controls; that is, systems defined by a Lagrangian and external forces depending on controls [2, 4]. These are considered as implicit systems defined by the Euler-Lagrange equations. Second, we analyze a quadratic optimal control problem for a descriptor system [17]. We point out the importance of these kinds of systems in engineering problems [18] and references therein. Finally, we include an Appendix where geometric features about Tulczyjew's operators, contact systems and the Euler-Lagrange equations for forced systems.

2 Skinner-Rusk unified formalism for non-autonomous systems

This formalism is a particular case of the unified formalism for field theories developed in [9] and also in [13]. See [7] for an alternative but equivalent approach, and [11] for an extension of this formalism to other kinds of more general time-dependent singular differential equations.

2.1 Previous results on non-autonomous Lagrangian and Hamiltonian systems

See, for instance, [10, 12, 16, 19, 21] for more details.

In the jet bundle description of non-autonomous dynamical systems, the configuration bundle is $\pi\colon E \longrightarrow \mathbb{R}$, where E is a $(n+1)$-dimensional differentiable manifold endowed with local coordinates (t, q^i), and $\mathbb{R}$ has as a global coordinate t. The jet bundle of local sections of π, $J^1\pi$, is the *velocity phase space* of the system, with natural coordinates (t, q^i, v^i), adapted to the bundle $\pi\colon E \longrightarrow \mathbb{R}$, and natural projections

$$\pi^1\colon J^1\pi \longrightarrow E \quad , \quad \bar{\pi}^1\colon J^1\pi \longrightarrow \mathbb{R}\,.$$

(In the case that $\pi\colon E \equiv \mathbb{R}\times Q \longrightarrow \mathbb{R}$, where Q is a n-dimensional differentiable manifold, then $J^1\pi \simeq \mathbb{R}\times \mathrm{T}Q$).

A Lagrangian density $\mathcal{L} \in \Omega^1(J^1\pi)$ is a $\bar{\pi}^1$-semibasic 1-form on $J^1\pi$, and it is usually written as $\mathcal{L} = L\,\mathrm{d}t$, where $L \in \mathrm{C}^\infty(J^1\pi)$ is the *Lagrangian function* determined by $\mathcal{L}$. Throughout this paper we denote by $\mathrm{d}t$ the volume form in $\mathbb{R}$, and its pull-backs to all the manifolds.

The *Poincaré-Cartan forms* associated with the Lagrangian density $\mathcal{L}$ are defined using the *vertical endomorphism* $\mathcal{V}$ of the bundle $J^1\pi$ (see [10, 22])

$$\Theta_{\mathcal{L}} = i(\mathcal{V})\mathrm{d}\mathcal{L} + \mathcal{L} \in \Omega^1(J^1\pi) \quad ; \quad \Omega_{\mathcal{L}} = -\mathrm{d}\Theta_{\mathcal{L}} \in \Omega^2(J^1\pi)\,.$$

A Lagrangian $\mathcal{L}$ is *regular* if $\Omega_{\mathcal{L}}$ has maximal rank; elsewhere $\mathcal{L}$ is singular. In natural coordinates we have $\mathcal{V} = (\mathrm{d}q^i - v^i\mathrm{d}t)\otimes \dfrac{\partial}{\partial v^i}\otimes\dfrac{\partial}{\partial t}$, and

$$\begin{aligned}
\Theta_{\mathcal{L}} &= \frac{\partial L}{\partial v^i}\mathrm{d}q^i - \left(\frac{\partial L}{\partial v^i}v^i - L\right)\mathrm{d}t \\
\Omega_{\mathcal{L}} &= -\frac{\partial^2 L}{\partial v^j\partial v^i}\mathrm{d}v^j\wedge\mathrm{d}q^i - \frac{\partial^2 L}{\partial q^j\partial v^i}\mathrm{d}q^j\wedge\mathrm{d}q^i \\
&\quad + \frac{\partial^2 L}{\partial v^j\partial v^i}v^i\mathrm{d}v^j\wedge\mathrm{d}t + \left(\frac{\partial^2 L}{\partial q^j\partial v^i}v^i - \frac{\partial L}{\partial q^j} + \frac{\partial^2 L}{\partial t\partial v^j}\right)\mathrm{d}q^j\wedge\mathrm{d}t\,.
\end{aligned}$$

The regularity condition is equivalent to $det\left(\dfrac{\partial^2 L}{\partial v^i\partial v^j}(\bar{y})\right)\neq 0$, for every $\bar{y}\in J^1\pi$. Geometrically, $\mathcal{L}$ is regular if and only if $(\Omega_{\mathcal{L}}, \mathrm{d}t)$ is a cosymplectic structure on $J^1\pi$. This means that $\Omega_{\mathcal{L}}$ and $\mathrm{d}t$ are closed and $\Omega_{\mathcal{L}}^n\wedge\mathrm{d}t$ is a volume form (see [15]).

The *Lagrangian problem* consists in finding sections $\phi\colon \mathbb{R} \longrightarrow E$ of π, which are characterized by the condition

$$(j^1\phi)^*\, i(X)\Omega_{\mathcal{L}} = 0 \quad , \quad \text{for every } X \in \mathfrak{X}(J^1\pi)$$

where $j^1\phi : \mathbb{R} \longrightarrow J^1\pi$ is the 1-jet extension of the section ϕ. In natural coordinates, if $\phi(t) = (t, \phi^i(t))$, this condition is equivalent to demanding that ϕ satisfies the *Euler-Lagrange equations*

$$\frac{\partial L}{\partial q^i}\Big|_{j^1\phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right)\Big|_{j^1\phi} = 0 \quad , \quad (\text{for } i = 1, \ldots, n) \tag{1}$$

where $j^1\phi(t) = (t, \phi^i(t), \dot{\phi}^i(t))$. Assuming that these sections are integral curves of vector fields in $J^1\pi$ the corresponding equations for these vector fields are

$$i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0 \quad , \quad i(X_{\mathcal{L}})\mathrm{d}t = 1 \tag{2}$$

where $X_{\mathcal{L}} \in \mathfrak{X}(J^1\pi)$ is holonomic (recall that a vector field in $J^1\pi$ is said to be holonomic, or also a *second order differential equation* (SODE for simplicity), if its integral curves are holonomic; that is, canonical liftings of sections $\varphi\colon \mathbb{R} \longrightarrow E$). In the regular case, there is a unique solution to these equations. In the singular case the existence of a solution is not assured, except perhaps on some submanifold (more generally, some subset) of $J^1\pi$, where the solution is not unique, in general.

Consider now the *extended momentum phase space* T^*E, and the *restricted momentum phase space* which is defined by $J^1\pi^* = \mathrm{T}^*E/\pi^*\mathrm{T}^*\mathbb{R}$. Local coordinates in these manifolds are (t, q^i, p, p_i) and (t, q^i, p_i), respectively. Then, the following natural projections are

$$\tau^1\colon J^1\pi^* \longrightarrow E \quad , \quad \bar{\tau}^1 = \pi \circ \tau^1\colon J^1\pi^* \longrightarrow \mathbb{R} \quad , \quad \mu\colon \mathrm{T}^*E \longrightarrow J^1\pi^* \quad , \quad p\colon \mathrm{T}^*E \longrightarrow \mathbb{R}\,.$$

Let $\Theta \in \Omega^1(\mathrm{T}^*E)$ and $\Omega = -\mathrm{d}\Theta \in \Omega^2(\mathrm{T}^*E)$ be the canonical forms of T^*E whose local expressions are

$$\Theta = p_i\mathrm{d}q^i + p\mathrm{d}t \quad , \quad \Omega = \mathrm{d}q^i \wedge \mathrm{d}p_i + \mathrm{d}t \wedge \mathrm{d}p\,.$$

(In the particular case $E = \mathbb{R} \times Q$, we have $\mathrm{T}^*E \simeq \mathbb{R} \times \mathbb{R}^* \times \mathrm{T}^*Q$, and $J^1\pi^* \simeq \mathbb{R} \times \mathrm{T}^*Q$ and introducing the projections $pr_1\colon \mathrm{T}^*(\mathbb{R} \times Q) \longrightarrow \mathbb{R} \times \mathbb{R}^*$, $pr_2\colon \mathrm{T}^*(\mathbb{R} \times Q) \longrightarrow \mathrm{T}^*Q$, we have $\Theta = pr_1^*\Theta_{\mathbb{R}} + pr_2^*\Theta_Q$ and $\Omega = pr_1^*\Omega_{\mathbb{R}} + pr_2^*\Omega_Q$; where $\Omega_{\mathbb{R}} = -\mathrm{d}\Theta_{\mathbb{R}} \in \Omega^2(\mathbb{R} \times \mathbb{R}^*)$ and $\Omega_Q = -\mathrm{d}\Theta_Q \in \Omega^2(\mathrm{T}^*Q)$ denote the natural symplectic forms of T^*Q and $\mathbb{R} \times \mathbb{R}^*$).

Being $\Theta_{\mathcal{L}} \in \Omega^1(J^1\pi)$ π^1-semibasic, we have a natural map $\widetilde{\mathcal{FL}}\colon J^1\pi \longrightarrow \mathrm{T}^*E$, given by

$$\widetilde{\mathcal{FL}}(\bar{y}) = \Theta_{\mathcal{L}}(\bar{y}) \tag{3}$$

which is called the *extended Legendre map* associated to the Lagrangian density $\mathcal{L}$. The *restricted Legendre map* is $\mathcal{FL} = \mu \circ \widetilde{\mathcal{FL}}\colon J^1\pi \longrightarrow J^1\pi^*$. Their local expressions are

$$\widetilde{\mathcal{FL}}^*t = t \quad , \quad \widetilde{\mathcal{FL}}^*q^i = q^i \quad , \quad \widetilde{\mathcal{FL}}^*p_i = \frac{\partial L}{\partial v^i} \quad , \quad \widetilde{\mathcal{FL}}^*p = L - v^i\frac{\partial L}{\partial v^i}$$

$$\mathcal{FL}^*t = t \quad , \quad \mathcal{FL}^*q^i = q^i \quad , \quad \mathcal{FL}^*p_i = \frac{\partial L}{\partial v^i}$$

or, in other words, $\widetilde{\mathcal{FL}}(t, q^i, \dot{q}^i) = (t, q^i, L - v^i\frac{\partial L}{\partial v^i}, \frac{\partial L}{\partial v^i})$ and $\mathcal{FL}(t, q^i, \dot{q}^i) = (t, q^i, \frac{\partial L}{\partial v^i})$. Moreover, we have $\widetilde{\mathcal{FL}}^*\Theta = \Theta_{\mathcal{L}}$, and $\widetilde{\mathcal{FL}}^*\Omega = \Omega_{\mathcal{L}}$.

The hyper-regular and regular cases

The Lagrangian $\mathcal{L}$ is regular if, and only if, $\mathcal{FL}$ is a local diffeomorphism. As a particular case, $\mathcal{L}$ is a *hyper-regular* Lagrangian if $\mathcal{FL}$ is a global diffeomorphism.

If $\mathcal{L}$ is a hyper-regular Lagrangian, then $\tilde{\mathcal{P}} = \widetilde{\mathcal{FL}}(J^1\pi)$ is a 1-codimensional, μ-transverse imbedded submanifold of T^*E, with natural imbedding $\tilde{\jmath}_0 \colon \tilde{\mathcal{P}} \hookrightarrow \mathrm{T}^*E$, which is diffeomorphic to $J^1\pi^*$. This diffeomorphism is the inverse of μ restricted to $\tilde{\mathcal{P}}$, and also coincides with the map $h = \widetilde{\mathcal{FL}} \circ \mathcal{FL}^{-1}$, when it is restricted onto its image (which is just $\tilde{\mathcal{P}}$). This map h is called a *Hamiltonian section*, and can be used to construct the *Hamilton-Cartan forms* in $J^1\pi^*$ by making

$$\Theta_h = h^*\Theta \in \mathit{\Omega}^1(J^1\pi^*) \quad , \quad \Omega_h = h^*\Omega \in \mathit{\Omega}^2(J^1\pi^*)\,.$$

Locally, the Hamiltonian section h is specified by $h(t, q^i, p_i) = (t, q^i, -H, p_i)$, where H is the local Hamiltonian function given by $H = p_i(F\mathcal{L}^{-1})^*v^i - (F\mathcal{L}^{-1})^*L$. The local expressions are

$$\Theta_h = p_i \mathrm{d}q^i - H\mathrm{d}t \quad , \quad \Omega_h = \mathrm{d}q^i \wedge \mathrm{d}p_i + \mathrm{d}H \wedge \mathrm{d}t\,.$$

Of course $\mathcal{FL}^*\Theta_h = \Theta_{\mathcal{L}}$, and $\mathcal{FL}^*\Omega_h = \Omega_{\mathcal{L}}$.

The *Hamiltonian problem* consists in finding sections of $\bar{\tau}^1$, $\psi \colon \mathbb{R} \longrightarrow J^1\pi^*$, which are characterized by the condition

$$\psi^*\, i(X)\Omega_h = 0 \quad , \quad \text{for every } X \in \mathfrak{X}(J^1\pi^*)\ .$$

This condition leads to the *Hamilton equations* which, if $\psi(t) = (t, q^i(t), p_i(t))$, in natural coordinates are

$$\frac{dq^i}{dt} = \left.\frac{\partial H}{\partial p_i}\right|_{\psi} \quad ; \quad \frac{dp_i}{dt} = -\left.\frac{\partial H}{\partial q^i}\right|_{\psi}\,.$$

Assuming that these sections are integral curves of vector fields $X_h \in \mathfrak{X}(J^1\pi^*)$, the corresponding equations for these vector fields are

$$i(X_h)\Omega_h = 0 \quad , \quad i(X_h)\mathrm{d}t = 1\,.$$

As a final remark, it can be proved that solutions to the Lagrangian and Hamiltonian problems are equivalent, in the sense that they are $\mathcal{FL}$-related; that is,

$$\psi = \mathcal{FL} \circ j^1\phi \quad ; \quad \mathrm{T}\mathcal{FL} \circ X_{\mathcal{L}} = X_h \circ \mathcal{FL}\,. \tag{4}$$

For regular, but not hyper-regular systems, the results are the same, but only locally on open neighbourhoods at every point, instead of $J^1\pi^*$.

The almost-regular case

A singular Lagrangian $\mathcal{L}$ is *almost-regular* if: $\mathcal{P} = \mathcal{FL}(J^1\pi)$ is a closed submanifold of $J^1\pi^*$ (let $\jmath \colon \mathcal{P} \hookrightarrow J^1\pi^*$ be natural imbedding), $\mathcal{FL}$ is a submersion onto its image, and for every $\bar{y} \in J^1\pi$, the fibres $\mathcal{FL}^{-1}(\mathcal{FL}(\bar{y}))$ are connected submanifolds of $J^1\pi$.

If $\mathcal{L}$ is an almost-regular Lagrangian, the submanifold $\mathcal{P}$ of $J^1\pi^*$ is a fibre bundle over E and M. In this case the μ-transverse submanifold $\tilde{\jmath} \colon \tilde{\mathcal{P}} \hookrightarrow \mathrm{T}^*E$ is diffeomorphic to $\mathcal{P}$. This diffeomorphism is denoted by $\tilde{\mu} \colon \tilde{\mathcal{P}} \longrightarrow \mathcal{P}$, and is just the restriction of the projection μ to $\tilde{\mathcal{P}}$. Then, taking the Hamiltonian section $\tilde{h} = \tilde{\jmath} \circ \tilde{\mu}^{-1}$, we define the forms

$$\Theta_h^0 = \tilde{h}^*\Theta \quad ; \quad \Omega_h^0 = \tilde{h}^*\Omega$$

which verify that $\mathcal{FL}_0^*\Theta_h^0 = \Theta_{\mathcal{L}}$ and $\mathcal{FL}_0^*\Omega_h^0 = \Omega_{\mathcal{L}}$ (where $\mathcal{FL}_0$ is the restriction map of $\mathcal{FL}$ onto $\mathcal{P}$). Then we have the following diagram

$$\begin{tikzcd}
 & \tilde{\mathcal{P}} \arrow[rr, "\tilde{\jmath}"'] \arrow[d, shift left, "\tilde{\mu}"] & & T^*E \arrow[d, "\mu"'] \\
J^1\pi \arrow[ur, "\widetilde{\mathcal{FL}_0}"] \arrow[r, "\mathcal{FL}_0"'] & \mathcal{P} \arrow[u, shift left, "\tilde{\mu}^{-1}"] \arrow[rr, "\jmath"'] \arrow[urr, "\tilde{h}"'] \arrow[dr, "\bar{\tau}_0^1"'] & & J^1\pi^* \arrow[dl, "\bar{\tau}^1"] \\
 & & \mathbb{R} &
\end{tikzcd}$$

Then, the Hamiltonian problem and the equations of motion are stated as in the hyper-regular case. Now, the existence of a solution to these equations is not assured, except perhaps on some submanifold of $\mathcal{P}$, where the solution is not unique, in general.

2.2 Unified formalism

We define the *extended jet-momentum bundle* $\mathcal{W}$ and the *restricted jet-momentum bundle* $\mathcal{W}_r$

$$\mathcal{W} = J^1\pi \times_E \mathrm{T}^*E \quad , \quad \mathcal{W}_r = J^1\pi \times_E J^1\pi^*$$

with natural coordinates (t, q^i, v^i, p, p_i) and (t, q^i, v^i, p_i), respectively. We have the natural submersions

$$\begin{array}{c}\rho_1 \colon \mathcal{W} \longrightarrow J^1\pi \ , \ \rho_2 \colon \mathcal{W} \longrightarrow \mathrm{T}^*E \ , \ \rho_E \colon \mathcal{W} \longrightarrow E \ , \ \rho_{\mathbb{R}} \colon \mathcal{W} \longrightarrow \mathbb{R} \\ \rho_1^r \colon \mathcal{W}_r \longrightarrow J^1\pi \ , \ \rho_2^r \colon \mathcal{W}_r \longrightarrow J^1\pi^* \ , \ \rho_E^r \colon \mathcal{W}_r \longrightarrow E \ , \ \rho_{\mathbb{R}}^r \colon \mathcal{W}_r \longrightarrow \mathbb{R}\,. \end{array} \tag{5}$$

Note that $\pi^1 \circ \rho_1 = \tau^1 \circ \mu \circ \rho_2 = \rho_E$. In addition, for $\bar{y} \in J^1\pi$, and $\mathbf{p} \in \mathrm{T}^*E$, there is also the natural projection

$$\begin{array}{cccc} \mu_{\mathcal{W}} & : & \mathcal{W} & \longrightarrow & \mathcal{W}_r \\ & & (\bar{y}, \mathbf{p}) & \mapsto & (\bar{y}, [\mathbf{p}]) \end{array}$$

where $[\mathbf{p}] = \mu(\mathbf{p}) \in J^1\pi^*$. The bundle $\mathcal{W}$ is endowed with the following canonical structures:

Definition 1 *1. The* coupling 1-form *in $\mathcal{W}$ is the $\rho_{\mathbb{R}}$-semibasic 1-form $\hat{\mathcal{C}} \in \mathit{\Omega}^1(\mathcal{W})$ defined as follows: for every $w = (j^1\phi(t), \alpha) \in \mathcal{W}$ (that is, $\alpha \in T^*_{\rho_E(w)}E$) and $V \in \mathrm{T}_w\mathcal{W}$, then*

$$\hat{\mathcal{C}}(V) = \alpha(\mathrm{T}_w(\phi \circ \rho_{\mathbb{R}})V)\,.$$

2. The canonical 1-form $\Theta_{\mathcal{W}} \in \mathit{\Omega}^1(\mathcal{W})$ *is the ρ_E-semibasic form defined by $\Theta_{\mathcal{W}} = \rho_2^*\Theta$.*
The canonical 2-form *is $\Omega_{\mathcal{W}} = -\mathrm{d}\Theta_{\mathcal{W}} = \rho_2^*\Omega \in \mathit{\Omega}^2(\mathcal{W})$.*

Being $\hat{\mathcal{C}}$ a $\rho_{\mathbb{R}}$-semibasic form, there is $\hat{C} \in \mathrm{C}^\infty(\mathcal{W})$ such that $\hat{\mathcal{C}} = \hat{C}\mathrm{d}t$. Note also that $\Omega_{\mathcal{W}}$ is degenerate, its kernel being the ρ_2-vertical vectors; then $(\mathcal{W}, \Omega_{\mathcal{W}})$ is a presymplectic manifold.

The local expressions for $\Theta_{\mathcal{W}}$, $\Omega_{\mathcal{W}}$, and $\hat{\mathcal{C}}$ are

$$\Theta_{\mathcal{W}} = p_i\mathrm{d}q^i + p\mathrm{d}t \quad , \quad \Omega_{\mathcal{W}} = -\mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}p \wedge \mathrm{d}t \quad , \quad \hat{\mathcal{C}} = (p + p_iv^i)\mathrm{d}t\,.$$

Given a Lagrangian density $\mathcal{L} \in \mathit{\Omega}^1(J^1\pi)$, we denote $\hat{\mathcal{L}} = \rho_1^*\mathcal{L} \in \mathit{\Omega}^1(\mathcal{W})$, and we can write $\hat{\mathcal{L}} = \hat{L}\mathrm{d}t$, with $\hat{L} = \rho_1^*L \in \mathrm{C}^\infty(\mathcal{W})$. We define a *Hamiltonian submanifold*

$$\mathcal{W}_0 = \{w \in \mathcal{W} \mid \hat{\mathcal{L}}(w) = \hat{\mathcal{C}}(w)\}\,.$$

So, $\mathcal{W}_0$ is the submanifold of $\mathcal{W}$ defined by the regular constraint function $\hat{C} - \hat{L} = 0$. Observe that this function is globally defined in $\mathcal{W}$, using the dynamical data and the geometry. In local coordinates this constraint function is

$$p + p_i v^i - \hat{L}(t, q^j, v^j) = 0 \tag{6}$$

and its meaning will be clear when we apply this formalism to Optimal Control problems (see Section 3.2). The natural imbedding is $\jmath_0 \colon \mathcal{W}_0 \hookrightarrow \mathcal{W}$, and we have the projections (submersions), see diagram (7):

$$\rho_1^0 \colon \mathcal{W}_0 \longrightarrow J^1\pi \ , \ \rho_2^0 \colon \mathcal{W}_0 \longrightarrow \mathrm{T}^*E \ , \ \rho_E^0 \colon \mathcal{W}_0 \longrightarrow E \ , \ \rho_{\mathbb{R}}^0 \colon \mathcal{W}_0 \longrightarrow \mathbb{R}$$

which are the restrictions to $\mathcal{W}_0$ of the projections (5), and

$$\hat{\rho}_2^0 = \mu \circ \rho_2^0 \colon \mathcal{W}_0 \longrightarrow J^1\pi^* \, .$$

Local coordinates in $\mathcal{W}_0$ are (t, q^i, v^i, p_i), and we have that

$$\begin{array}{lcl} \rho_1^0(t, q^i, v^i, p_i) = (t, q^i, v^i) & , & \jmath_0(t, q^i, v^i, p_i) = (t, q^i, v^i, L - v^i p_i, p_i) \\ \hat{\rho}_2^0(t, q^i, v^i, p_i) = (t, q^i, p_i) & , & \rho_2^0(t, q^i, v^i, p_i) = (t, q^i, L - v^i p_i, p_i) \, . \end{array}$$

Proposition 1 *$\mathcal{W}_0$ is a 1-codimensional $\mu_{\mathcal{W}}$-transverse submanifold of $\mathcal{W}$, which is diffeomorphic to $\mathcal{W}_r$.*

(*Proof*) For every $(\bar{y}, \mathbf{p}) \in \mathcal{W}_0$, we have $L(\bar{y}) \equiv \hat{L}(\bar{y}, \mathbf{p}) = \hat{C}(\bar{y}, \mathbf{p})$, and

$$(\mu_{\mathcal{W}} \circ \jmath_0)(\bar{y}, \mathbf{p}) = \mu_{\mathcal{W}}(\bar{y}, \mathbf{p}) = (\bar{y}, \mu(\mathbf{p})) \, .$$

First, $\mu_{\mathcal{W}} \circ \jmath_0$ is injective: let $(\bar{y}_1, \mathbf{p}_1), (\bar{y}_2, \mathbf{p}_2) \in \mathcal{W}_0$, then we have

$$(\mu_{\mathcal{W}} \circ \jmath_0)(\bar{y}_1, \mathbf{p}_1) = (\mu_{\mathcal{W}} \circ \jmath_0)(\bar{y}_2, \mathbf{p}_2) \ \Rightarrow \ (\bar{y}_1, \mu(\mathbf{p}_1)) = (\bar{y}_2, \mu(\mathbf{p}_2)) \ \Rightarrow \ \bar{y}_1 = \bar{y}_2 \ , \ \mu(\mathbf{p}_1) = \mu(\mathbf{p}_2)$$

hence $L(\bar{y}_1) = L(\bar{y}_2) = \hat{C}(\bar{y}_1, \mathbf{p}_1) = \hat{C}(\bar{y}_2, \mathbf{p}_2)$. In a local chart, the third equality gives

$$p(\mathbf{p}_1) + p_i(\mathbf{p}_1) v^i(\bar{y}_1) = p(\mathbf{p}_2) + p_i(\mathbf{p}_2) v^i(\bar{y}_2)$$

but $\mu(\mathbf{p}_1) = \mu(\mathbf{p}_2)$ implies that

$$p_i(\mathbf{p}_1) = p_i([\mathbf{p}_1]) = p_i([\mathbf{p}_2]) = p_i(\mathbf{p}_2)$$

therefore $p(\mathbf{p}_1) = p(\mathbf{p}_2)$ and hence $\mathbf{p}_1 = \mathbf{p}_2$.

Second, $\mu_{\mathcal{W}} \circ \jmath_0$ is onto, then, if $(\bar{y}, [\mathbf{p}]) \in \mathcal{W}_r$, there exists $(\bar{y}, \mathbf{q}) \in \jmath_0(\mathcal{W}_0)$ such that $[\mathbf{q}] = [\mathbf{p}]$. In fact, it suffices to take $[\mathbf{q}]$ such that, in a local chart of $J^1\pi \times_E \mathrm{T}^*E = \mathcal{W}$

$$p_i(\mathbf{q}) = p_i([\mathbf{p}]) \ , \ p(\mathbf{q}) = L(\bar{y}) - p_i([\mathbf{p}]) v^i(\bar{y}) \, .$$

Finally, since $\mathcal{W}_0$ is defined by the constraint function $\hat{C} - \hat{L}$ and, as $\ker \mu_{\mathcal{W}*} = \left\{ \dfrac{\partial}{\partial p} \right\}$ locally and $\dfrac{\partial}{\partial p}(\hat{C} - \hat{L}) = 1$, then $\mathcal{W}_0$ is $\mu_{\mathcal{W}}$-transversal. ■

As a consequence of this result, the submanifold $\mathcal{W}_0$ induces a section of the projection $\mu_{\mathcal{W}}$,

$$\hat{h} \colon \mathcal{W}_r \longrightarrow \mathcal{W} \, .$$

Locally, $\hat{h}$ is specified by giving the local *Hamiltonian function* $\hat{H} = -\hat{L} + p_i v^i$; that is, $\hat{h}(t, q^i, v^i, p_i) = (t, q^i, v^i, -\hat{H}, p_i)$. In this sense, $\hat{h}$ is said to be a *Hamiltonian section* of $\mu_{\mathcal{W}}$.

So we have the following diagram

$$
\begin{tikzcd}
 & J^1\pi & \\
\mathcal{W}_0 \arrow[ur, "\rho_1^0"] \arrow[r, "\jmath_0"] \arrow[dr, "\rho_2^0"] \arrow[ddr, "\hat{\rho}_2^0"'] & \mathcal{W} \arrow[u, "\rho_1"] \arrow[r, "\mu_{\mathcal{W}}"] \arrow[d, "\rho_2"] & \mathcal{W}_r \arrow[ul, "\rho_1^r"'] \arrow[dl, "\rho_2 \circ \hat{h}"'] \arrow[ddl, "\rho_2^r"] \\
 & T^*E \arrow[d, "\mu"] & \\
 & J^1\pi^* &
\end{tikzcd}
\tag{7}
$$

Remark: Observe that, from the Hamiltonian $\mu_{\mathcal{W}}$-section $\hat{h}\colon \mathcal{W}_r \longrightarrow \mathcal{W}$ in the extended unified formalism, we can recover the Hamiltonian μ-section $\tilde{h} = \tilde{\jmath} \circ \tilde{\mu}^{-1}\colon \mathcal{P} \longrightarrow \mathrm{T}^*E$ in the standard Hamiltonian formalism assuming that $\mathcal{L}$ is almost-regular. In fact, given $[\mathbf{p}] \in J^1\pi^*$, the section $\hat{h}$ maps every point $(\bar{y}, [\mathbf{p}]) \in (\rho_2^r)^{-1}([\mathbf{p}])$ into $\rho_2^{-1}[\rho_2(\hat{h}(\bar{y}, [\mathbf{p}]))]$. Now, the crucial point is the projectability of the local function $\hat{H}$ by ρ_2. However, $\dfrac{\partial}{\partial v^i}$ being a local basis for $\ker \rho_{2*}$, $\hat{H}$ is ρ_2-projectable if, and only if, $p_i = \dfrac{\partial L}{\partial v^i}$, and this condition is fulfilled when $[\mathbf{p}] \in \mathcal{P} = \mathrm{Im}\,\mathcal{FL} \subset J^1\pi^*$, which implies that $\rho_2[\hat{h}((\rho_2^r)^{-1}([\mathbf{p}]))] \in \tilde{\mathcal{P}} = \mathrm{Im}\,\widetilde{\mathcal{FL}} \subset \mathrm{T}^*E$. Then, the Hamiltonian section $\tilde{h}$ is defined as

$$
\tilde{h}([\mathbf{p}]) = (\rho_2 \circ \hat{h})[(\rho_2^r)^{-1}(\jmath([\mathbf{p}]))] = (\tilde{\jmath} \circ \tilde{\mu}^{-1})([\mathbf{p}]) \ , \ \text{for every } [\mathbf{p}] \in \mathcal{P}.
$$

So we have the diagram

$$
\begin{tikzcd}
\tilde{\mathcal{P}} \arrow[r, "\tilde{\jmath}"'] \arrow[d, shift left, "\tilde{\mu}"] & T^*E \arrow[d, "\mu"] & \mathcal{W} \arrow[l, "\rho_2"'] \\
\mathcal{P} \arrow[u, shift left, "\tilde{\mu}^{-1}"] \arrow[ur, "\tilde{h}"'] \arrow[r, "\jmath"'] & J^1\pi^* & \mathcal{W}_r \arrow[l, "\rho_2^r"] \arrow[u, "\hat{h}"']
\end{tikzcd}
$$

For (hyper) regular systems this diagram is the same with $\mathcal{P} = \mathrm{Im}\,\mathcal{FL} = J^1\pi^*$.

Finally, we can define the forms

$$
\Theta_0 = \jmath_0^*\Theta_{\mathcal{W}} = \rho_2^{0*}\Theta \in \mathit{\Omega}^1(\mathcal{W}_0) \quad , \quad \Omega_0 = \jmath_0^*\Omega_{\mathcal{W}} = \rho_2^{0*}\Omega \in \mathit{\Omega}^2(\mathcal{W}_0)
$$

with local expressions

$$
\Theta_0 = (L - p_i v^i)\mathrm{d}t + p_i \mathrm{d}q^i \quad , \quad \Omega_0 = \mathrm{d}(p_i v^i - L) \wedge \mathrm{d}t - \mathrm{d}p_i \wedge \mathrm{d}q^i \tag{8}
$$

and we have obtained a presymplectic Hamiltonian system $(\mathcal{W}_0, \Omega_0)$, or equivalently $(\mathcal{W}_r, \Omega_r)$, with $\Omega_r = \hat{h}^*\Omega_0$.

2.3 The dynamical equations for sections

Now we are going to establish the dynamical problem for the system $(\mathcal{W}_0, \Omega_0)$ which, as a consequence of the diffeomorphism stated in Proposition 1, is equivalent to making it for the system $(\mathcal{W}_r, \Omega_r)$.

The *Lagrange-Hamiltonian problem* associated with the system $(\mathcal{W}_0, \Omega_0)$ consists in finding sections of $\rho^0_{\mathbb{R}}$, $\psi_0\colon \mathbb{R} \longrightarrow \mathcal{W}_0$, which are characterized by the condition

$$\psi_0^*\, i(Y_0)\Omega_0 = 0 \quad , \quad \text{for every } Y_0 \in \mathfrak{X}(\mathcal{W}_0) \ . \tag{9}$$

This equation gives different kinds of information, depending on the type of the vector fields Y_0 involved. In particular, using vector fields Y_0 which are $\hat{\rho}^0_2$-vertical, denoted by $\mathfrak{X}^{\mathrm{V}(\hat{\rho}^0_2)}(\mathcal{W}_0)$, we have:

Lemma 1 *If $Y_0 \in \mathfrak{X}^{\mathrm{V}(\hat{\rho}^0_2)}(\mathcal{W}_0)$, then $i(Y_0)\Omega_0$ is $\rho^0_{\mathbb{R}}$-semibasic.*

(*Proof*) A simple calculation in coordinates leads to this result. In fact, taking $\left\{\dfrac{\partial}{\partial v^i}\right\}$ as a local basis for the $\hat{\rho}^0_2$-vertical vector fields, and bearing in mind (8) we obtain

$$i\left(\frac{\partial}{\partial v^i}\right)\Omega_0 = \left(p_i - \frac{\partial L}{\partial v^i}\right) \mathrm{d}t$$

which are obviously $\rho^0_{\mathbb{R}}$-semibasic forms. ■

As an immediate consequence, when $Y_0 \in \mathfrak{X}^{\mathrm{V}(\hat{\rho}^0_2)}(\mathcal{W}_0)$, condition (9) does not depend on the derivatives of ψ_0: it is a pointwise (algebraic) condition. We can define the submanifold

$$\mathcal{W}_1 = \{(\bar{y}, \mathbf{p}) \in \mathcal{W}_0 \ | \ i(V_0)(\Omega_0)_{(\bar{y},\mathbf{p})} = 0, \ \text{for every } V_0 \in \mathrm{V}_{(\bar{y},\mathbf{p})}(\hat{\rho}^0_2)\}$$

where $\mathrm{V}(\hat{\rho}^0_2)$ denotes the $\hat{\rho}^0_2$-vertical vectors. $\mathcal{W}_1$ is called the *first constraint submanifold* of the Hamiltonian pre-multisymplectic system $(\mathcal{W}_0, \Omega_0)$, as every section ψ_0 solution to (9) must take values in $\mathcal{W}_1$. We denote by $\jmath_1\colon \mathcal{W}_1 \hookrightarrow \mathcal{W}_0$ the natural embedding.

Locally, $\mathcal{W}_1$ is defined in $\mathcal{W}_0$ by the constraints $p_i = \dfrac{\partial L}{\partial v^i}$. Moreover:

Proposition 2 *$\mathcal{W}_1$ is the graph of $\widetilde{\mathcal{FL}}$; that is, $\mathcal{W}_1 = \{(\bar{y}, \widetilde{\mathcal{FL}}(\bar{y})) \in \mathcal{W} \ | \ \bar{y} \in J^1\pi\}$.*

(*Proof*) Consider $\bar{y} \in J^1\pi$, let $\phi\colon \mathbb{R} \longrightarrow E$ be a representative of $\bar{y}$, and $\mathbf{p} = \widetilde{\mathcal{FL}}(\bar{y})$. For every $U \in \mathrm{T}_{\bar{\pi}^1(\bar{y})}\mathbb{R}$, consider $V = \mathrm{T}_{\bar{\pi}^1(\bar{y})}\phi(U)$ and its canonical lifting $\bar{V} = \mathrm{T}_{\bar{\pi}^1(\bar{y})}j^1\phi(U)$. From the definition of the extended Legendre map (3) we have $(\mathrm{T}_{\bar{y}}\pi^1)^*(\widetilde{\mathcal{FL}}(\bar{y})) = (\Theta_{\mathcal{L}})_{\bar{y}}$, then

$$i(\bar{V})[(\mathrm{T}_{\bar{y}}\pi^1)^*(\widetilde{\mathcal{FL}}(\bar{y}))] = i(\bar{V})(\Theta_{\mathcal{L}})_{\bar{y}} \, .$$

Furthermore, as $\mathbf{p} = \widetilde{\mathcal{FL}}(\bar{y})$, we also have that

$$\begin{aligned} i(\bar{V})[(\mathrm{T}_{\bar{y}}\pi^1)^*(\widetilde{\mathcal{FL}}(\bar{y}))] &= i(\mathrm{T}_{\bar{\pi}^1(\bar{y})}j^1\phi(U))[(\mathrm{T}_{\bar{y}}\pi^1)^*\mathbf{p}] = i((\mathrm{T}_{\bar{y}}\pi^1)_*(\mathrm{T}_{\bar{\pi}^1(\bar{y})}j^1\phi(U)))\mathbf{p} \\ &= i(\mathrm{T}_{\bar{\pi}^1(\bar{y})}\phi(U))\mathbf{p} = i(V)\mathbf{p} \, . \end{aligned}$$

Therefore we obtain

$$i(U)(\phi^*\mathbf{p}) = i(U)[(j^1\phi)^*(\Theta_{\mathcal{L}})_{\bar{y}}]$$

and bearing in mind the definition of the coupling form $\mathcal{C}$, this condition becomes

$$i(U)(\hat{\mathcal{C}}(\bar{y}, \mathbf{p})) = i(U)[(j^1\phi)^*\Theta_{\mathcal{L}})_{\bar{y}}] \, .$$

Since it holds for every $U \in \mathrm{T}_{\bar{\pi}^1(\bar{y})}\mathbb{R}$, we conclude that $\hat{\mathcal{C}}(\bar{y}, \mathbf{p}) = [(j^1\phi)^*\Theta_{\mathcal{L}}]_{\bar{y}}$, or equivalently, $\hat{\mathcal{C}}(\bar{y}, \mathbf{p}) = \hat{L}(\bar{y}, \mathbf{p})$, where we have made use of the fact that $\Theta_{\mathcal{L}}$ is the sum of the Lagrangian density $\mathcal{L}$ and a contact form $i(\mathcal{V})\mathrm{d}\mathcal{L}$ (vanishing by pull-back of lifted sections). This is the condition defining $\mathcal{W}_0$, and thus we have proved that $(\bar{y}, \widetilde{\mathcal{FL}}(\bar{y})) \in \mathcal{W}_0$, for every $\bar{y} \in J^1\pi$; that is, $\operatorname{graph}\widetilde{\mathcal{FL}} \subset \mathcal{W}_0$. Furthermore, $\operatorname{graph}\widetilde{\mathcal{FL}}$ and $\mathcal{W}_1$ are defined as subsets of $\mathcal{W}_0$ by the same local conditions: $p_i - \dfrac{\partial L}{\partial v^i} = 0$. So we conclude that $\operatorname{graph}\widetilde{\mathcal{FL}} = \mathcal{W}_1$. ■

As $\mathcal{W}_1$ is the graph of $\widetilde{\mathcal{FL}}$, it is diffeomorphic to $J^1\pi$. Every section $\psi_0 \colon \mathbb{R} \longrightarrow \mathcal{W}_0$ is of the form $\psi_0 = (\psi_{\mathcal{L}}, \psi_{\mathcal{H}})$, with $\psi_{\mathcal{L}} = \rho_1^0 \circ \psi_0 \colon \mathbb{R} \longrightarrow J^1\pi$, and if ψ_0 takes values in $\mathcal{W}_1$ then $\psi_{\mathcal{H}} = \widetilde{\mathcal{FL}} \circ \psi_{\mathcal{L}} \colon \mathbb{R} \longrightarrow \mathrm{T}^*E$. In this way every constraint, differential equation, etc. in the unified formalism can be translated to the Lagrangian or the Hamiltonian formalisms by restriction to the first or the second factors of the product bundle.

However, as was pointed out before, the geometric condition (9) in $\mathcal{W}_0$, which can be solved only for sections $\psi_0 \colon \mathbb{R} \longrightarrow \mathcal{W}_1 \subset \mathcal{W}_0$, is stronger than the Lagrangian condition $\psi_{\mathcal{L}}^*\, i(Z)\Omega_{\mathcal{L}} = 0$, (for every $Z \in \mathfrak{X}(J^1\pi)$) in $J^1\pi$, which can be translated to $\mathcal{W}_1$ by the natural diffeomorphism between them. The reason is that, as ρ_1^0 is a submersion, and $\mathcal{W}_1$ is a ρ_1^0-transversal submanifold of $\mathcal{W}_0$ (as a consequence of Proposition 2), we have the splitting $\jmath_1^*\mathrm{T}\mathcal{W}_0 = \mathrm{T}\mathcal{W}_1 \oplus_{\mathcal{W}_1} \jmath_1^*\mathrm{V}(\rho_1^0)$, $\jmath_1 \colon \mathcal{W}_1 \hookrightarrow \mathcal{W}_0$ being the natural embedding. Therefore the additional information comes from the ρ_1^0-vertical vectors, and is just the holonomic condition. In fact:

Theorem 1 *Let $\psi_0 \colon \mathbb{R} \longrightarrow \mathcal{W}_0$ be a section fulfilling equation (9), $\psi_0 = (\psi_{\mathcal{L}}, \psi_{\mathcal{H}}) = (\psi_{\mathcal{L}}, \widetilde{\mathcal{FL}} \circ \psi_{\mathcal{L}})$, where $\psi_{\mathcal{L}} = \rho_1^0 \circ \psi_0$. Then:*

1. *$\psi_{\mathcal{L}}$ is the canonical lift of the projected section $\phi = \rho_E^0 \circ \psi_0 \colon \mathbb{R} \longrightarrow E$ (that is, $\psi_{\mathcal{L}}$ is a holonomic section).*
2. *The section $\psi_{\mathcal{L}} = j^1\phi$ is a solution to the Lagrangian problem, and the section $\mu \circ \psi_{\mathcal{H}} = \mu \circ \widetilde{\mathcal{FL}} \circ \psi_{\mathcal{L}} = \mathcal{FL} \circ j^1\phi$ is a solution to the Hamiltonian problem.*

Conversely, for every section $\phi \colon \mathbb{R} \longrightarrow E$ such that $j^1\phi$ is a solution to the Lagrangian problem (and hence $\mathcal{FL} \circ j^1\phi$ is a solution to the Hamiltonian problem) we have that the section $\psi_0 = (j^1\phi, \widetilde{\mathcal{FL}} \circ j^1\phi)$, is a solution to (9).

$$
\begin{tikzcd}
 & \mathcal{W} \arrow[ddl, "\rho_1"'] \arrow[dr, "\rho_2"] & & \\
 & \mathcal{W}_0 \arrow[u, "\jmath_0"] \arrow[r, "\rho_2^0"] \arrow[dl, "\rho_1^0"] & \mathrm{T}^*E & \\
J^1\pi \arrow[dr, "\pi^1"] & \mathcal{W}_1 \arrow[u, "\jmath_1"] \arrow[l, "\rho_1^1"'] \arrow[r, "\rho_2^1"] \arrow[d, "\rho_E^1"] & J^1\pi^* \arrow[dl, "\tau^1"] & \mathrm{T}^*E \\
 & E & & \\
 & \mathbb{R} \arrow[u, "\phi"] \arrow[uu, bend right, "\psi_1"'] \arrow[uuu, bend right, "\psi_0"'] \arrow[uul, "\psi_{\mathcal{L}} = j^1\phi"] \arrow[uurr, "\psi_{\mathcal{H}} = \widetilde{\mathcal{FL}} \circ j^1\phi"'] & &
\end{tikzcd}
$$

(*Proof*)

1. Taking $\left\{ \dfrac{\partial}{\partial p_i} \right\}$ as a local basis for the ρ_1^0-vertical vector fields:

$$i\left(\frac{\partial}{\partial p_i}\right)\Omega_0 = v^i \mathrm{d}t - \mathrm{d}q^i$$

so that for a section ψ_0 we have

$$0 = \psi_0^* \left[i\left(\frac{\partial}{\partial p_i}\right)\Omega_0 \right] = \left(v^i - \frac{\partial q^i}{\partial t} \right) \mathrm{d}t$$

and thus the holonomy condition appears naturally within the unified formalism, and it is not necessary to impose it by hand to ψ_0. Thus we have that $\psi_0 = \left(t, q^i, \dfrac{dq^i}{dt}, \dfrac{\partial L}{\partial v^i} \right)$, since ψ_0 takes values in $\mathcal{W}_1$, and hence it is of the form $\psi_0 = (j^1\phi, \widetilde{\mathcal{FL}} \circ j^1\phi)$, for $\phi = (t, q^i) = \rho^0_{_E} \circ \psi_0$.

2. Since sections $\psi_0 \colon \mathbb{R} \longrightarrow \mathcal{W}_0$ solution to (9) take values in $\mathcal{W}_1$, we can identify them with sections $\psi_1 \colon \mathbb{R} \longrightarrow \mathcal{W}_1$. These sections ψ_1 verify, in particular, that $\psi_1^* \, i(Y_1)\Omega_1 = 0$ holds for every $Y_1 \in \mathfrak{X}(\mathcal{W}_1)$. Obviously $\psi_0 = \jmath_1 \circ \psi_1$. Moreover, as $\mathcal{W}_1$ is the graph of $\widetilde{\mathcal{FL}}$, denoting by $\rho_1^1 = \rho_1^0 \circ \jmath_1 \colon \mathcal{W}_1 \longrightarrow J^1\pi$ the diffeomorphism which identifies $\mathcal{W}_1$ with $J^1\pi$, if we define $\Omega_1 = \jmath_1^*\Omega_0$, we have that $\Omega_1 = \rho_1^{1*}\Omega_{\mathcal{L}}$. In fact; as $(\rho_1^1)^{-1}(\bar{y}) = (\bar{y}, \widetilde{\mathcal{FL}}(\bar{y}))$, for every $\bar{y} \in J^1\pi$, then $(\rho_2^0 \circ \jmath_1 \circ (\rho_1^1)^{-1})(\bar{y}) = \widetilde{\mathcal{FL}}(\bar{y}) \in \mathrm{T}^*E$, and hence

$$\Omega_{\mathcal{L}} = (\rho_2^0 \circ \jmath_1 \circ (\rho_1^1)^{-1})^*\Omega = [((\rho_1^1)^{-1})^* \circ \jmath_1^* \circ \rho_2^{0*}]\Omega = [((\rho_1^1)^{-1})^* \circ \jmath_1^*]\Omega_0 = ((\rho_1^1)^{-1})^*\Omega_1 \, .$$

Now, let $X \in \mathfrak{X}(J^1\pi)$. We have

$$\begin{aligned}
(j^1\phi)^* \, i(X)\Omega_{\mathcal{L}} &= (\rho_1^0 \circ \psi_0)^* \, i(X)\Omega_{\mathcal{L}} = (\rho_1^0 \circ \jmath_1 \circ \psi_1)^* \, i(X)\Omega_{\mathcal{L}} \\
&= (\rho_1^1 \circ \psi_1)^* \, i(X)\Omega_{\mathcal{L}} = \psi_1^* \, i((\rho_1^1)_*^{-1}X)(\rho_1^{1*}\Omega_{\mathcal{L}}) = \psi_1^* \, i(Y_1)\Omega_1 \\
&= \psi_1^* \, i(Y_1)(\jmath_1^*\Omega_0) = (\psi_1^* \circ \jmath_1^*) \, i(Y_0)\Omega_0 = \psi_0^* \, i(Y_0)\Omega_0
\end{aligned} \tag{10}$$

where $Y_0 \in \mathfrak{X}(\mathcal{W}_0)$ is such that $Y_0 = \jmath_{1*}Y_1$. But as $\psi_0^* \, i(Y_0)\Omega_0 = 0$, for every $Y_0 \in \mathfrak{X}(\mathcal{W}_0)$, then we conclude that $(j^1\phi)^* \, i(X)\Omega_{\mathcal{L}} = 0$, for every $X \in \mathfrak{X}(J^1\pi)$.

Conversely, let $j^1\phi \colon \mathbb{R} \longrightarrow J^1\pi$ such that $(j^1\phi)^* \, i(X)\Omega_{\mathcal{L}} = 0$, for every $X \in \mathfrak{X}(J^1\pi)$, and define $\psi_0 \colon \mathbb{R} \longrightarrow \mathcal{W}_0$ as $\psi_0 = (j^1\phi, \widetilde{\mathcal{FL}} \circ j^1\phi)$ (observe that ψ_0 takes its values in $\mathcal{W}_1$). Taking into account that, on the points of $\mathcal{W}_1$, every $Y_0 \in \mathfrak{X}(\mathcal{W}_0)$ splits into $Y_0 = Y_0^1 + Y_0^2$, with $Y_0^1 \in \mathfrak{X}(\mathcal{W}_0)$ tangent to $\mathcal{W}_1$, and $Y_0^2 \in \mathfrak{X}^{\mathrm{V}(\rho_1^0)}(\mathcal{W}_0)$, we have that

$$\psi_0^* \, i(Y_0)\Omega_0 = \psi_0^* \, i(Y_0^1)\Omega_0 + \psi_0^* \, i(Y_0^2)\Omega_0 = 0$$

since for Y_0^1 the same reasoning as in (10) leads to

$$\psi_0^* \, i(Y_0^1)\Omega_0 = (j^1\phi)^* \, i(X_0^1)\Omega_{\mathcal{L}} = 0$$

(where $X_0^1 = (\rho_1^1)_* Y_0^1$), and for Y_0^2, also following the same reasoning as in (10), a local calculus gives

$$\psi_0^* \, i(Y_0^2)\Omega_0 = (j^1\phi)^* \left[\left(f_i(x) \left(v_\alpha^A - \frac{\partial q^i}{\partial x^\alpha} \right) \right) \mathrm{d}t \right] = 0$$

since $j^1\phi$ is a holonomic section and $Y_0^2 = f_i \dfrac{\partial}{\partial p_i}$.

The result for the sections $\psi_{\mathcal{H}} = \widetilde{\mathcal{FL}} \circ j^1\phi$ is a direct consequence of the first equivalence relations (4). ■

Remark: The results in this section can also be recovered in coordinates taking an arbitrary local vector field $Y_0 = f\dfrac{\partial}{\partial t} + f^i\dfrac{\partial}{\partial q^i} + g^i\dfrac{\partial}{\partial v^i} + h_i\dfrac{\partial}{\partial p_i} \in \mathfrak{X}(\mathcal{W}_0)$, then

$$\begin{aligned} i(Y_0)\Omega_0 &= -f\left(p_i \mathrm{d}v^i + v^i \mathrm{d}p_i - \frac{\partial L}{\partial q^i}\mathrm{d}q^i - \frac{\partial L}{\partial v^i}\mathrm{d}v^i\right) \\ &\quad -f^i\left(\frac{\partial L}{\partial q^i}\mathrm{d}t + \mathrm{d}p_i\right) + g^i\left(p_i - \frac{\partial L}{\partial v^i}\right)\mathrm{d}t + h_i(v^i\mathrm{d}t - \mathrm{d}q^i) \end{aligned}$$

and, for a section ψ_0 fulfilling (9),

$$0 = \psi_0^*\, i(Y_0)\Omega_0 = \left[f^i\left(\frac{dp_i}{dt} - \frac{\partial L}{\partial q^i}\right) + g^i\left(p_i - \frac{\partial L}{\partial v^i}\right) + h_i\left(v^i - \frac{dq^i}{dt}\right)\right]\mathrm{d}t \tag{11}$$

reproduces the holonomy condition, the restricted Legendre map (that is, the definition of the momenta), and the Euler-Lagrange equations. The coefficient of the component f vanishes as a consequence of the last equations.

Summarizing, the equation (9) gives different kinds of information, depending on the type of verticality of the vector fields Y_0 involved. In particular we have obtained equations of three different classes:

1. Algebraic (not differential) equations, in coordinates $p_i = \dfrac{\partial L}{\partial v^i}$, which determine a subset $\mathcal{W}_1$ of $\mathcal{W}_0$, where the sections solution must take their values. These can be called *primary Hamiltonian constraints*, and in fact they generate, by $\hat{\rho}_2^0$ projection, the primary constraints of the Hamiltonian formalism for singular Lagrangians, i.e., the image of the Legendre transformation, $\mathcal{FL}(J^1\pi) \subset J^1\pi^*$.
2. The holonomic differential equations, in coordinates $v^i = \dfrac{dq^i}{dt}$, forcing the sections solution ψ_0 to be lifting of π-sections. This property reflects the fact that the geometric condition in the unified formalism is stronger than the usual one in the Lagrangian formalism.
3. The classical Euler-Lagrange equations, in coordinates
$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial^2 L}{\partial v^j \partial v^i}\frac{d^2 q^j}{dt^2} + \frac{\partial^2 L}{\partial q^j \partial v^i}\frac{dq^j}{dt} + \frac{\partial^2 L}{\partial t \partial v^i} = \frac{\partial L}{\partial q^i} \tag{12}$$
which are obtained from $\dfrac{dp_i}{dt} = \dfrac{\partial L}{\partial q^i}$, using the previous equations.

2.4 The dynamical equations for vector fields

Proposition 3 *The problem of finding sections solutions to (9) is equivalent to finding the integral curves of a vector field $X_0 \in \mathfrak{X}(\mathcal{W}_0)$, which is tangent to $\mathcal{W}_1$ and satisfies that*

$$i(X_0)\Omega_0 = 0 \quad , \quad i(X_0)\mathrm{d}t = 1\,. \tag{13}$$

(*Proof*) In a natural chart in $\mathcal{W}_0$, the local expression of a vector field $X_0 \in \mathfrak{X}(\mathcal{W}_0)$ is

$$X_0 = f\frac{\partial}{\partial t} + F^i\frac{\partial}{\partial q^i} + G^i\frac{\partial}{\partial v^i} + H_i\frac{\partial}{\partial p_i}\,.$$

Then, the second equation (13) leads to $f = 1$, and the first gives

$$\text{coefficients in } \mathrm{d}p_i : \quad F^i = v^i \tag{14}$$

$$\text{coefficients in } \mathrm{d}v^i : \quad p_i = \frac{\partial L}{\partial v^i} \tag{15}$$

$$\text{coefficients in } \mathrm{d}q^i : \quad H_i = \frac{\partial L}{\partial q^i} \tag{16}$$

$$\text{coefficients in } \mathrm{d}t : \quad -F^i \frac{\partial L}{\partial q^i} + G^i \left(p_i - \frac{\partial L}{\partial v^i} \right) + H_i v^i = 0 \, . \tag{17}$$

Now, if $\psi_0 = (t, q^i(t), v^i(t), p_i(t))$ is an integral curve of X_0, we have that $F^i = \dfrac{dq^i}{dt}$, $G^i = \dfrac{dv^i}{dt}$, $H_i = \dfrac{dp_i}{dt}$, and then (see equation (11)):

- Equations (14) are the holonomy condition.
- The algebraic equations (15) are the compatibility conditions defining $\mathcal{W}_1$.
- Using (14) and (15), equations (16) are the Euler-Lagrange equations (12).
- Taking into account (14) and (16), equation (17) holds identically.

Observe that the condition that X_0 (if it exists) must be tangent to $\mathcal{W}_1$ holds also identically from the above equations, since

$$0 = X_0 \left(p_i - \frac{\partial L}{\partial v^i} \right) = -\frac{\partial^2 L}{\partial v^i \partial v^j} G^i - \frac{\partial^2 L}{\partial t \partial v^j} - \frac{\partial^2 L}{\partial q^i \partial v^j} v^i + \frac{\partial L}{\partial q^j} \qquad (\text{on } \mathcal{W}_1)$$

are the Euler-Lagrange equations again. Observe that, if L is a regular Lagrangian (that is, the matrix $\left(\dfrac{\partial^2 L}{\partial v^i \partial v^j}(\bar{y}) \right)$ is regular), these Euler -Lagrange equations allow us to determine the functions $G^i = \dfrac{dv^i}{dt}$. If L is a singular Lagrangian, then a constraint algorithm must be used in order to obtain a submanifold (if it exists) where consistent solutions exist. ■

Now, the equivalence of the unified formalism with the Lagrangian and Hamiltonian formalisms can be recovered as follows:

Theorem 2 *Let X_0 be a vector field in $\mathcal{W}_0$ which is the solution to the equations (13). Then the vector field $X_\mathcal{L} \in \mathfrak{X}(J^1\pi)$ defined by*

$$X_\mathcal{L} \circ \rho_1^0 = \mathrm{T}\rho_1^0 \circ X_0$$

is a holonomic vector field solution to the equations (2).

Conversely, every holonomic vector field solution to the equations (2) can be recovered in this way from a vector field $X_0 \in \mathfrak{X}_{\mathcal{W}_1}(\mathcal{W}_0)$.

(*Proof*) Let X_0 be a vector field on $\mathcal{W}_0$, which is a solution to (13). As sections $\psi_0 \colon \mathbb{R} \longrightarrow \mathcal{W}_0$ solution to the geometric equation (9) must take value in $\mathcal{W}_1$, then X_0 can be identified with a vector field $X_1 \colon \mathcal{W}_0 \longrightarrow \mathrm{T}\mathcal{W}_1$ (i.e., $\mathrm{T}\jmath_1 \circ X_1 = X_0|_{\mathcal{W}_1}$), and hence there exists $X_\mathcal{L} \colon J^1\pi \longrightarrow \mathrm{T}(J^1\pi)$ such that $X_1 = \mathrm{T}(\rho_1^1)^{-1} \circ X_\mathcal{L} \in \mathfrak{X}(\mathcal{W}_1)$. Therefore, as a consequence of the item 1 in Theorem 1, for every section ψ_0 solution to (9), there exists $X_\mathcal{L}^0 \in \mathfrak{X}(j^1\phi(\mathbb{R}))$ such that $\mathrm{T}\jmath_\phi \circ X^0_{\mathcal{L}} = X_{\mathcal{L}}|_{j^1\phi(\mathbb{R})}$, where $\jmath_\phi \colon j^1\phi(\mathbb{R}) \longrightarrow E$ is the natural imbedding. So, $X_{\mathcal{L}}$ is $\bar{\pi}^1$-transversal and holonomic. Then, bearing in mind that $\jmath_1^*\Omega_0 = \rho_1^{1*}\Omega_{\mathcal{L}}$, we have

$$\jmath_1^*\, i(X_0)\Omega_0 = i(X_1)(\jmath_1^*\Omega_0) = i(X_1)(\rho_1^{1*}\Omega_{\mathcal{L}}) = \rho_1^{1*}\, i(X_{\mathcal{L}})\Omega_{\mathcal{L}}$$

then $i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$ because $i(X_0)\Omega_0 = 0$. A similar reasoning leads us to prove that, if $i(X_0)\mathrm{d}t = 1$, then $i(X_{\mathcal{L}})\mathrm{d}t = 1$.

Conversely, given a holonomic vector field $X_{\mathcal{L}}$, from $i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$, and taking into account the above chain of equalities, we obtain that $i(X_0)\Omega_0 \in [\mathfrak{X}(\mathcal{W}_1)]^0$ (the annihilator of $\mathfrak{X}(\mathcal{W}_1)$). Moreover, $X_{\mathcal{L}}$ being holonomic, X_0 is holonomic, and then the extra condition $i(Y_0)\, i(X_0)\Omega_0 = 0$ is also fulfilled for every $Y_0 \in \mathfrak{X}^{\mathrm{V}(\rho_1^0)}(\mathcal{W}_0)$. Thus, remembering that $\jmath_1^*\mathrm{T}\mathcal{W}_0 = \mathrm{T}\mathcal{W}_1 \oplus_{\mathcal{W}_1} \jmath_1^*\mathrm{V}(\rho_1^0)$, we conclude that $i(X_0)\Omega_0 = 0$. To prove that if $i(X_{\mathcal{L}})\mathrm{d}t = 1$, then $i(X_0)\mathrm{d}t = 1$ is trivial. ∎

Finally, the Hamiltonian formalism is recovered using the second equivalence relations (4). The proof for the almost-regular case follows in a straightforward way.

3 Optimal control theory

3.1 General features

In this section we consider non-autonomous optimal control systems. This class of systems are determined by the *state equations*, which are a set of differential equations

$$\dot{q}^i = \mathcal{F}^i(t, q^j(t), u^a(t))\,,\ 1 \leq i \leq n\,, \tag{18}$$

where t is time, q^j denote the state variables and u^a, $1 \leq a \leq m$, the control inputs of the system that must be determined. Prescribing initial conditions of the state variables and fixing control inputs we know completely the trajectory of the state variables $q^j(t)$ (in the sequel, all the functions are assumed to be at least C^2). The objective is the following:

Statement 1 *(Non-autonomous optimal control problem) Find a C^2-piecewise smooth curve $\gamma(t) = (t, q^j(t), u^a(t))$ and $T \in \mathbb{R}^+$ satisfying the conditions for the state variables at time 0 and T, the control equations (18); and minimizing the functional $\mathcal{J}(\gamma) = \int_0^T \mathbb{L}(t, q^j(t), u^a(t))\,\mathrm{d}t$.*

In a global description, we have a fiber bundle structure $\pi^C \colon C \longrightarrow E$ and $\pi \colon E \longrightarrow \mathbb{R}$, where E is equipped with natural coordinates (t, q^i) and C is the bundle of controls, with coordinates (t, q^i, u^a).

The state equations can be geometrically described as a smooth map $\mathcal{F} : C \longrightarrow J^1\pi$ such that it makes commutative the following diagram

$$\begin{tikzcd}
C \arrow[rr, "\mathcal{F}"] \arrow[dr, "\pi^C"] \arrow[ddr, "\bar{\pi}^C"'] & & J^1\pi \arrow[dl, "\pi^1"'] \arrow[ddl, "\bar{\pi}^1"] \\
& E \arrow[d, "\pi"] & \\
& \mathbb{R} &
\end{tikzcd}$$

which means that $\mathcal{F}$ is a jet field along π^C and also along $\bar{\pi}^C$. Locally we have $\mathcal{F}(t, q^i, u^a) = (t, q^i, \mathcal{F}^i(t, q^i, u^a))$.

A necessary condition for the solutions of such problems are provided by *Pontryaguin's Maximum Principle*.

Theorem 3 (Pontryaguin's Maximum Principle): *If a curve* $\gamma : [0,T] \to C$, $\gamma(t) = (t, q^i(t), u^a(t))$, *with* $\gamma(0)$ *and* $\gamma(T)$ *fixed, is an optimal trajectory, then there exist functions* $p_i(t)$, $1 \leq i \leq n$, *verifying:*

$$\begin{aligned} \frac{dq^i}{dt} &= \frac{\partial \mathcal{H}}{\partial p_i}(t, q^i(t), u^a(t), p_i(t)) & (19) \\ \frac{dp_i}{dt} &= -\frac{\partial \mathcal{H}}{\partial q^i}(t, q^i(t), u^a(t), p_i(t)) & (20) \\ \mathcal{H}(t, q^i(t), u^a(t), p_i(t)) &= \max_{u^a} \mathcal{H}(t, q^i(t), u^a, p_i(t)), \quad t \in [0,T] & (21) \end{aligned}$$

where

$$\mathcal{H}(t, q^i, u^a, p_i) = p_j \mathcal{F}^j(t, q^i, u^a) + p_0 \mathbb{L}(t, q^i, u^a)$$

and $p_0 \in \{-1, 0\}$.

When we are looking for extremal trajectories, which are those satisfying the necessary conditions of Theorem 3, condition (21) is usually replaced by the weaker condition

$$\varphi_a \equiv \frac{\partial \mathcal{H}}{\partial u^a} = 0, \quad 1 \leq a \leq m \ .$$

In this weaker form, the Maximum Principle only applies to optimal trajectories with optimal controls interior to the control fibres.

Remark: An extremal trajectory is called *normal* if $p_0 = -1$ and *abnormal* if $p_0 = 0$. For the sake of simplicity, we only consider normal extremal trajectories, but the necessary conditions for abnormal extremals can also be characterized geometrically using the formalism given in Section 2. Hence, from now on we will take $p_0 = -1$.

An optimal control problem is said to be *regular* if the matrix

$$\left(\frac{\partial \varphi_a}{\partial u^b} \right) = \left(\frac{\partial^2 \mathcal{H}}{\partial u^a \partial u^b} \right) \tag{22}$$

has maximal rank.

3.2 Unified geometric framework for optimal control theory

Geometrically, we will assume that an *optimal control system* is determined by the pair $(\mathbf{L}, \mathcal{F})$, where $\mathbf{L} \in \Omega^1(C)$ is a $\bar{\pi}^C$-semibasic 1-form, then $\mathbf{L} = \mathbb{L} dt$, with $\mathbb{L} \in \mathrm{C}^\infty(C)$ representing the cost function; and $\mathcal{F}$ is the jet field introduced in the above section.

The graph of the mapping $\mathcal{F}$, $\mathrm{Graph}\,\mathcal{F}$, is a subset of $C \times_E J^1\pi$ and allows us to define the *extended control-jet-momentum bundle* and the *restricted control-jet-momentum bundle*, respectively:

$$\mathcal{W}^{\mathcal{F}} = \mathrm{Graph}\,\mathcal{F} \times_E T^*E \quad , \quad \mathcal{W}_r^{\mathcal{F}} = \mathrm{Graph}\,\mathcal{F} \times_E J^1\pi^*$$

which are submanifolds of $C \times_E \mathcal{W} = C \times_E J^1\pi \times_E \mathrm{T}^*E$ and $C \times_E \mathcal{W}_r = C \times_E J^1\pi \times_E J^1\pi^*$, respectively.

In $\mathcal{W}^{\mathcal{F}}$ and $\mathcal{W}_r^{\mathcal{F}}$ we have natural coordinates (t, q^i, u^a, p, p_i) and (t, q^i, u^a, p_i), respectively. We have the following natural projections (submersions), see diagram (27):

$$\begin{array}{rclcrclcrcl} \mu_{\mathcal{W}^{\mathcal{F}}} \colon \mathcal{W}^{\mathcal{F}} & \longrightarrow & \mathcal{W}_r^{\mathcal{F}} & , & \rho_1^{\mathcal{F}} \colon \mathcal{W}^{\mathcal{F}} & \longrightarrow & C & , & \rho_2^{\mathcal{F}} \colon \mathcal{W}^{\mathcal{F}} & \longrightarrow & \mathrm{T}^*E \\ \rho_E^{\mathcal{F}} \colon \mathcal{W}^{\mathcal{F}} & \longrightarrow & E & , & \rho_{\mathbb{R}}^{\mathcal{F}} \colon \mathcal{W}^{\mathcal{F}} & \longrightarrow & \mathbb{R} & , & \rho_1^{r\mathcal{F}} \colon \mathcal{W}_r^{\mathcal{F}} & \longrightarrow & C \\ \rho_2^{r\mathcal{F}} \colon \mathcal{W}_r^{\mathcal{F}} & \longrightarrow & J^1\pi^* & , & \rho_E^{r\mathcal{F}} \colon \mathcal{W}_r^{\mathcal{F}} & \longrightarrow & E & , & \rho_{\mathbb{R}}^{r\mathcal{F}} \colon \mathcal{W}_r^{\mathcal{F}} & \longrightarrow & \mathbb{R}\,. \end{array} \tag{23}$$

In addition we also have the immersions, see diagram (24):

$$\begin{array}{rcl} i^{\mathcal{F}} & : & \mathcal{W}^{\mathcal{F}} \hookrightarrow C \times_E \mathcal{W}\,, \quad i^{\mathcal{F}}(t, q^i, u^a, p, p_i) = (t, q^i, u^a, \mathcal{F}^i(t, q^j, u^b,), p, p_i) \\ i_r^{\mathcal{F}} & : & \mathcal{W}_r^{\mathcal{F}} \hookrightarrow C \times_E \mathcal{W}_r\,, \quad i_r^{\mathcal{F}}(t, q^i, u^a, p_i) = (t, q^i, u^a, \mathcal{F}^i(t, q^j, u^b), p_i)\ , \end{array}$$

and taking the natural projection

$$\sigma_{\mathcal{W}} \colon C \times_E \mathcal{W} \longrightarrow \mathcal{W}$$

we can construct the pullback of the coupling 1-form $\hat{\mathcal{C}}$ and of the forms $\Theta_{\mathcal{W}}$ and $\Omega_{\mathcal{W}}$ to $\mathcal{W}^{\mathcal{F}}$:

$$\mathcal{C}_{\mathcal{W}^{\mathcal{F}}} = (\sigma_{\mathcal{W}} \circ i^{\mathcal{F}})^*\hat{\mathcal{C}} \quad , \quad \Theta_{\mathcal{W}^{\mathcal{F}}} = (\sigma_{\mathcal{W}} \circ i^{\mathcal{F}})^*\Theta_{\mathcal{W}} \quad , \quad \Omega_{\mathcal{W}^{\mathcal{F}}} = (\sigma_{\mathcal{W}} \circ i^{\mathcal{F}})^*\Omega_{\mathcal{W}} = (\rho_2^{\mathcal{F}})^*\Omega\,,$$

see Definition 1, whose local expressions are:

$$\Theta_{\mathcal{W}^{\mathcal{F}}} = p_i \mathrm{d}q^i + p\mathrm{d}t \quad , \quad \Omega_{\mathcal{W}^{\mathcal{F}}} = -\mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}p \wedge \mathrm{d}t, \quad , \quad \mathcal{C}_{\mathcal{W}^{\mathcal{F}}} = (p + p_i\mathcal{F}^i(t, q^j, u^a))\mathrm{d}t\,.$$

Hence, we can draw the diagram

$$\begin{tikzcd}
C \times_E \mathcal{W} \arrow[rrrr, "\mathrm{Id} \times \mu_{\mathcal{W}}"] \arrow[dddr, "\sigma_{\mathcal{W}}"'] & & & & C \times_E \mathcal{W}_r \arrow[dddl, "\sigma_{\mathcal{W}_r}"] \\
& \mathcal{W}^{\mathcal{F}} \arrow[ul, "i^{\mathcal{F}}"] \arrow[rr, "\mu_{\mathcal{W}^{\mathcal{F}}}"] \arrow[dr, "\rho_2^{\mathcal{F}}"'] & & \mathcal{W}_r^{\mathcal{F}} \arrow[ur, "i_r^{\mathcal{F}}"'] & \\
& & \mathrm{T}^*E & & \\
& \mathcal{W} \arrow[ur, "\rho_2"] \arrow[rr, "\mu_{\mathcal{W}}"] & & \mathcal{W}_r &
\end{tikzcd} \tag{24}$$

Furthermore we can define the unique function $H_{\mathcal{W}^{\mathcal{F}}} \colon \mathcal{W}^{\mathcal{F}} \longrightarrow \mathbb{R}$ by the condition

$$\mathcal{C}_{\mathcal{W}^{\mathcal{F}}} - (\rho_1^{\mathcal{F}})^*\mathbf{L} = H_{\mathcal{W}^{\mathcal{F}}}\mathrm{d}t\,.$$

This function $H_{\mathcal{W}^{\mathcal{F}}}$ is locally described as

$$H_{\mathcal{W}^{\mathcal{F}}}(t, q^i, u^a, p, p_i) = p + p_i\mathcal{F}^i(t, q^j, u^a) - \mathbb{L}(t, q^j, u^a)\,; \tag{25}$$

compare this expression with (6). This is the natural Pontryaguin Hamiltonian function, which vanishes since we are considering a free-time problem.

Let $\mathcal{W}_0^{\mathcal{F}}$ be the submanifold of $\mathcal{W}^{\mathcal{F}}$ defined by the vanishing of $H_{\mathcal{W}^{\mathcal{F}}}$; that is,

$$\mathcal{W}_0^{\mathcal{F}} = \{w \in \mathcal{W}^{\mathcal{F}} \mid H_{\mathcal{W}^{\mathcal{F}}}(w) = 0\}\,.$$

In local coordinates, $\mathcal{W}_0^{\mathcal{F}}$ is given by the constraint

$$p + p_i \mathcal{F}^i(t, q^j, u^a) - \mathbb{L}(t, q^j, u^a) = 0$$

and an obvious set of coordinates in $\mathcal{W}_0^{\mathcal{F}}$ is (t, q^i, u^a, p_i). We denote by $\jmath_0^{\mathcal{F}} \colon \mathcal{W}_0^{\mathcal{F}} \longrightarrow \mathcal{W}^{\mathcal{F}}$ the natural embedding; in local coordinates,

$$\jmath_0^{\mathcal{F}}(t, q^i, u^a, p_i) = (t, q^i, u^a, \mathbb{L}(t, q^j, u^b) - p_i \mathcal{F}^i(t, q^j, u^b), p_j)$$

and we also have the projections (submersions)

$$\begin{array}{lclcl} \rho_1^{0\mathcal{F}} \colon \mathcal{W}_0^{\mathcal{F}} \longrightarrow C & , & \rho_E^{0\mathcal{F}} \colon \mathcal{W}_0^{\mathcal{F}} \longrightarrow E & , & \rho_{\mathbb{R}}^{0\mathcal{F}} \colon \mathcal{W}_0^{\mathcal{F}} \longrightarrow \mathbb{R} \\ \rho_2^{0\mathcal{F}} \colon \mathcal{W}_0^{\mathcal{F}} \longrightarrow \mathrm{T}^*E & , & \hat{\rho}_2^{0\mathcal{F}} \colon \mathcal{W}_0^{\mathcal{F}} \longrightarrow J^1\pi^* & , & \end{array}$$

which are the restrictions to $\mathcal{W}_0^{\mathcal{F}}$ of some of the projections (23), see diagram (27).

In a similar way to Proposition 1, we may prove the following:

Proposition 4 *$\mathcal{W}_0^{\mathcal{F}}$ is a 1-codimensional $\mu_{\mathcal{W}^{\mathcal{F}}}$-transverse submanifold of $\mathcal{W}^{\mathcal{F}}$, diffeomorphic to $\mathcal{W}_r^{\mathcal{F}}$.*

As a consequence, the submanifold $\mathcal{W}_0^{\mathcal{F}}$ induces a section of the projection $\mu_{\mathcal{W}^{\mathcal{F}}}$,

$$\hat{h}^{\mathcal{F}} \colon \mathcal{W}_r^{\mathcal{F}} \longrightarrow \mathcal{W}^{\mathcal{F}} \ . \tag{26}$$

Locally, $\hat{h}^{\mathcal{F}}$ is specified by giving the local *Hamiltonian function* $\hat{H}^{\mathcal{F}} = p_j \mathcal{F}^j - \mathbb{L}$; that is, $\hat{h}^{\mathcal{F}}(t, q^i, u^a, p_i) = (t, q^i, u^a, p = -\hat{H}^{\mathcal{F}}, p_i)$. The map $\hat{h}^{\mathcal{F}}$ is said to be a *Hamiltonian section* of $\mu_{\mathcal{W}^{\mathcal{F}}}$.

Thus, we can draw the diagram

$$\begin{tikzcd}
 & J^1\pi \arrow[dl, "\pi^1"'] \arrow[dr, "\bar{\pi}^1"] & \\
E & C \arrow[l, "\pi^C"'] \arrow[r, "\bar{\pi}^C"] \arrow[u, "\mathcal{F}"] & \mathbb{R} \\
\mathcal{W}_0^{\mathcal{F}} \arrow[u, "\rho_E^{0\mathcal{F}}"] \arrow[ur, "\rho_1^{0\mathcal{F}}"] \arrow[r, "\jmath_0^{\mathcal{F}}"] \arrow[dr, "\rho_2^{0\mathcal{F}}"] \arrow[ddr, "\hat{\rho}_2^{0\mathcal{F}}"'] & \mathcal{W}^{\mathcal{F}} \arrow[ul, "\rho_E^{\mathcal{F}}"] \arrow[u, "\rho_1^{\mathcal{F}}"] \arrow[ur, "\rho_{\mathbb{R}}^{\mathcal{F}}"] \arrow[r, "\mu_{\mathcal{W}^{\mathcal{F}}}"] \arrow[d, "\rho_2^{\mathcal{F}}"] & \mathcal{W}_r^{\mathcal{F}} \arrow[ul, "\rho_1^{r\mathcal{F}}"] \arrow[u, "\rho_{\mathbb{R}}^{r\mathcal{F}}"'] \arrow[dl, "\rho_2^{r\mathcal{F}} \circ \hat{h}^{\mathcal{F}}"'] \arrow[ddl, "\rho_2^{r\mathcal{F}}"] \\
 & T^*E \arrow[d, "\mu"] & \\
 & J^1\pi^* &
\end{tikzcd} \tag{27}$$

Finally we define the forms

$$\Theta_{\mathcal{W}_0^{\mathcal{F}}} = (\jmath_0^{\mathcal{F}})^*\Theta_{\mathcal{W}^{\mathcal{F}}} \quad , \quad \Omega_{\mathcal{W}_0^{\mathcal{F}}} = (\jmath_0^{\mathcal{F}})^*\Omega_{\mathcal{W}^{\mathcal{F}}}$$

with local expressions

$$\Theta_{\mathcal{W}_0^{\mathcal{F}}} = p_i \mathrm{d}q^i + (\mathbb{L} - p_i\mathcal{F}^i)\mathrm{d}t \quad , \quad \Omega_{\mathcal{W}_0^{\mathcal{F}}} = -\mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}(\mathbb{L} - p_i\mathcal{F}^i) \wedge \mathrm{d}t\,.$$

3.3 Optimal Control equations

Now we are going to establish the dynamical problem for the system $(\mathcal{W}_0^{\mathcal{F}}, \Omega_{\mathcal{W}_0^{\mathcal{F}}})$, thus obtaining a geometrical version of the weak form of the Maximum Principle.

Theorem 4 *If $\gamma(t) = (t, q^i(t), u^a(t))$ is a solution to the regular optimal control problem given by $(\mathbf{L}, \mathcal{F})$, then there exists an integral curve of a vector field $Z \in \mathfrak{X}(\mathcal{W}_0^{\mathcal{F}})$, whose projection to C is $\gamma(t)$, and such that Z is a solution to the equations*

$$i(Z)\Omega_{\mathcal{W}_0^{\mathcal{F}}} = 0 \quad , \quad i(Z)\mathrm{d}t = 1\ , \tag{28}$$

in a submanifold of $\mathcal{W}_0^{\mathcal{F}}$, which is given by the constraint algorithm.

(*Proof*) Locally, we have

$$Z = f\frac{\partial}{\partial t} + A^i\frac{\partial}{\partial q^i} + B^a\frac{\partial}{\partial u^a} + C_i\frac{\partial}{\partial p_i}$$

where f, A^i, B^a, C_i are unknown functions in $\mathcal{W}_0^{\mathcal{F}}$. Then, the second equation (28) leads to $f = 1$, and from the first we obtain that

$$\text{coefficients in } \mathrm{d}p_i : \quad \mathcal{F}^i - A^i = 0 \tag{29}$$

$$\text{coefficients in } \mathrm{d}u^a : \quad \frac{\partial\mathbb{L}}{\partial u^a} - p_j\frac{\partial\mathcal{F}^j}{\partial u^a} = 0 \tag{30}$$

$$\text{coefficients in } \mathrm{d}q^i : \quad \frac{\partial\mathbb{L}}{\partial q^i} - p_j\frac{\partial\mathcal{F}^j}{\partial q^i} - C_i = 0 \tag{31}$$

$$\text{coefficients in } \mathrm{d}t \ : \quad -A^i\frac{\partial\mathbb{L}}{\partial q^i} + A^i p_j\frac{\partial\mathcal{F}^j}{\partial q^i} - B^a\frac{\partial\mathbb{L}}{\partial u^a} + B^a p_j\frac{\partial\mathcal{F}^j}{\partial u^a} + C_i\mathcal{F}^i = 0\,. \tag{32}$$

Now, if $\psi_0^{\mathcal{F}} = (t, q^i(t), u^a(t), p_i(t))$ is an integral curve of Z, we have that $A^i = \dfrac{dq^i}{dt}$, $B^a = \dfrac{du^a}{dt}$, $C_i = \dfrac{dp_i}{dt}$. Then, considering the *Pontryaguin Hamiltonian function* $\mathcal{H}(t, q^i, u^a, p_i) = -\mathbb{L} + p_i\mathcal{F}^i(t, q^j, u^b)$, we have that:

- From (29) we deduce that $A^i = \mathcal{F}^i$; that is, $\dfrac{dq^i}{dt} = \dfrac{\partial\mathcal{H}}{\partial p_i}$, which are the equations (19).
- Equations (30) determine a new set of constraints

$$\varphi_a = \frac{\partial\mathbb{L}}{\partial u^a} - p_j\frac{\partial\mathcal{F}^j}{\partial u^a} = \frac{\partial\mathcal{H}}{\partial u^a} = 0$$

  which are assumed to define the new constraint submanifold $\mathcal{W}_1^{\mathcal{F}}$ of $\mathcal{W}_0^{\mathcal{F}}$. We denote by $\jmath_1^{\mathcal{F}} : \mathcal{W}_1^{\mathcal{F}} \hookrightarrow \mathcal{W}_0^{\mathcal{F}}$ the natural embedding.

- From (31) we completely determine the functions $C_i = \dfrac{dp_i}{dt} = -\dfrac{\partial \mathcal{H}}{\partial q^i}$; which are the equations (20).
- Finally, using (29), (31) and (30) it is easy to prove that equations (32) hold identically.

Furthermore Z must be tangent to $\mathcal{W}_1^{\mathcal{F}}$, that is,

$$Z(\varphi_a) = Z\left(\frac{\partial \mathcal{H}}{\partial u^a}\right) = 0 \qquad (\text{on } \mathcal{W}_1^{\mathcal{F}})$$

or, in other words,

$$0 = \frac{\partial^2 \mathcal{H}}{\partial t \partial u^a} + \mathcal{F}^i \frac{\partial^2 \mathcal{H}}{\partial q^i \partial u^a} + B^b \frac{\partial^2 \mathcal{H}}{\partial u^b \partial u^a} - \frac{\partial \mathcal{H}}{\partial q^i}\frac{\partial^2 \mathcal{H}}{\partial p_i \partial u^a} \qquad (\text{on } \mathcal{W}_1^{\mathcal{F}})\,. \tag{33}$$

However, as the optimal control problem is regular, the matrix $\dfrac{\partial^2 \mathcal{H}}{\partial u^b \partial u^a}$ has maximal rank. Then the system of equations (33) determines all the coefficients B^b.

Once the vector field Z is determined, we consider an integral curve that projects onto γ through $\rho_1^{\mathcal{F}}$. ■

Remark: In fact, the second equation of (28) could be relaxed to the condition

$$i(Z)\mathrm{d}t \neq 0\ ,$$

which determines vector fields transversal to π whose integral curves are equivalent to those obtained above, with arbitrary reparametrization.

Note that, using the implicit function theorem on the equations $\varphi_a = 0$, we get the functions $u^a = u^a(q,p,t)$. Therefore, for regular control problems, we can choose local coordinates (t, q^i, p_i) on $\mathcal{W}_1^{\mathcal{F}}$, and $\mathcal{H}|_{\mathcal{W}_1^{\mathcal{F}}}$ is locally a function of these coordinates.

If the control problem is not regular, then one has to implement a constraint algorithm to obtain a final constraint submanifold $\mathcal{W}_f^{\mathcal{F}}$ (if it exists) where the vector field Z is tangent (see, for instance, [8]).

Let $\jmath_1 \colon \mathcal{W}_1^{\mathcal{F}} \to \mathcal{W}_0^{\mathcal{F}}$ be the natural embedding, the form $\Omega_{\mathcal{W}_1^{\mathcal{F}}} = (\jmath_1^{\mathcal{F}})^*\Omega_{\mathcal{W}_0^{\mathcal{F}}}$ is locally written as

$$\Omega_{\mathcal{W}_1^{\mathcal{F}}} = -\mathrm{d}\mathcal{H}|_{\mathcal{W}_1^{\mathcal{F}}} \wedge \mathrm{d}t - \mathrm{d}p_i \wedge \mathrm{d}q^i\,.$$

Hence, for optimal control problems, taking into account the regularity of the matrix (22), we have the following:

Proposition 5 *If the optimal control problem is regular, then* $(\mathcal{W}_1^{\mathcal{F}}, \Omega_{\mathcal{W}_1^{\mathcal{F}}}, \mathrm{d}t)$ *is a cosymplectic manifold.*

4 Implicit optimal control problems

4.1 Unified geometric framework for implicit optimal control problems

The formalism presented in Section 3.2 is valid for a more general class of optimal control problems not previously considered from a geometric perspective: optimal control problems whose state equations are *implicit*, that is,

$$\Psi^\alpha(t,q,\dot q,u) = 0\ ,\ 1 \leq \alpha \leq s\ ,\ \text{with } \mathrm{d}\Psi^1 \wedge \ldots \wedge \mathrm{d}\Psi^s \neq 0\,. \tag{34}$$

From a more geometric point of view, we may interpret Equations (34) as constraint functions determining a submanifold M_C of $C \times_E J^1\pi$, with natural embedding $\jmath^{M_C} \colon M_C \hookrightarrow C \times_E J^1\pi$. We will also assume that $(\pi^C \times \pi^1) \circ \jmath^{M_C} \colon M_C \longrightarrow E$ is a surjective submersion.

In this situation, the techniques presented in the previous section are still valid. Now the implicit optimal control system is determined by the data $(\mathbf{L}, M_C)$, where $\mathbf{L} \in \Omega^1(M_C)$ is a semibasic form with respect to the projection $\tau^{M_C} \colon M_C \longrightarrow \mathbb{R}$, and hence it can be written as $\mathbf{L} = \mathbb{L}\mathrm{d}t$, for some $\mathbb{L} \in \mathrm{C}^\infty(M_C)$. First define the *extended control-jet-momentum manifold* and the *restricted control-jet-momentum manifold*

$$\mathcal{W}^{M_C} = M_C \times_E T^*E \quad , \quad \mathcal{W}_r^{M_C} = M_C \times_E J^1\pi^*$$

which are submanifolds of $C \times_E \mathcal{W} = C \times_E J^1\pi \times_E \mathrm{T}^*E$ and $C \times_E \mathcal{W}_r = C \times_E J^1\pi \times_E J^1\pi^*$, respectively.

We have the canonical immersions (embeddings)

$$i^{M_C} \colon \mathcal{W}^{M_C} \hookrightarrow C \times_E \mathcal{W} \quad , \quad i_r^{M_C} \colon \mathcal{W}_r^{M_C} \hookrightarrow C \times_E \mathcal{W}_r \; .$$

So we can draw the following diagram

$$\begin{tikzcd}
C \times_E \mathcal{W} \arrow[rrrr, "\mathrm{Id} \times \mu_{\mathcal{W}}"] \arrow[dddr, "\sigma_{\mathcal{W}}"'] & & & & C \times_E \mathcal{W}_r \arrow[dddl, "\sigma_{\mathcal{W}_r}"] \\
& & M_C & & \\
& \mathcal{W}^{M_C} \arrow[uul, "i^{M_C}"] \arrow[ur, "\rho_1^{M_C}"] \arrow[rr, "\mu_{\mathcal{W}^{M_C}}"] & & \mathcal{W}_r^{M_C} \arrow[ul, "\rho_1^{rM_C}"'] \arrow[uur, "i_r^{M_C}"'] & \\
& \mathcal{W} \arrow[rr, "\mu_{\mathcal{W}}"] & & \mathcal{W}_r &
\end{tikzcd} \tag{35}$$

Furthermore we also have the canonical projections (submersions)

$$\begin{array}{lllll}
\mu_{\mathcal{W}^{M_C}} \colon \mathcal{W}^{M_C} \longrightarrow \mathcal{W}_r^{M_C} & , & \rho_1^{M_C} \colon \mathcal{W}^{M_C} \longrightarrow M_C & , & \rho_2^{M_C} \colon \mathcal{W}^{M_C} \longrightarrow \mathrm{T}^*E \\
\rho_E^{M_C} \colon \mathcal{W}^{M_C} \longrightarrow E & , & \rho_{\mathbb{R}}^{M_C} \colon \mathcal{W}^{M_C} \longrightarrow \mathbb{R} & , & \rho_1^{rM_C} \colon \mathcal{W}_r^{M_C} \longrightarrow M_C \\
\rho_2^{rM_C} \colon \mathcal{W}_r^{M_C} \longrightarrow J^1\pi^* & , & \rho_E^{rM_C} \colon \mathcal{W}_r^{M_C} \longrightarrow E & , & \rho_{\mathbb{R}}^{rM_C} \colon \mathcal{W}_r^{M_C} \longrightarrow \mathbb{R} \; .
\end{array}$$

Now, consider the pullback of the coupling 1-form $\hat{\mathcal{C}}$ and the forms $\sigma_{\mathcal{W}}^*\Theta_{\mathcal{W}}$ and $\sigma_{\mathcal{W}}^*\Omega_{\mathcal{W}}$ to $\mathcal{W}^{M_C}$; that is

$$\mathcal{C}_{\mathcal{W}^{M_C}} = (\sigma_{\mathcal{W}} \circ i^{M_C})^*\hat{\mathcal{C}} \; , \; \Theta_{\mathcal{W}^{M_C}} = (\sigma_{\mathcal{W}} \circ i^{M_C})^*\Theta_{\mathcal{W}} \; , \; \Omega_{\mathcal{W}^{M_C}} = (\sigma_{\mathcal{W}} \circ i^{M_C})^*\Omega_{\mathcal{W}} \; ,$$

and denote by $\hat{C} \in \mathrm{C}^\infty(\mathcal{W}^{M_C})$ the unique function such that $\mathcal{C}_{\mathcal{W}^{M_C}} = \hat{C}\mathrm{d}t$. Finally, let $H_{\mathcal{W}^{M_C}} \colon \mathcal{W}^{M_C} \longrightarrow \mathbb{R}$ be the unique function such that $\mathcal{C}_{\mathcal{W}^{M_C}} - (\rho_1^{M_C})^*\mathbf{L} = H_{\mathcal{W}^{M_C}}\mathrm{d}t$. Observe that $H_{\mathcal{W}^{M_C}} = \hat{C} - \hat{\mathbb{L}}$, where $\hat{\mathbb{L}} = (\rho_1^{M_C})^*\mathbb{L}$, and remember that $H_{\mathcal{W}^{M_C}}$ is the Pontryaguin Hamiltonian function, see (25).

Let $\mathcal{W}_0^{M_C}$ be the submanifold of $\mathcal{W}^{M_C}$ defined by the vanishing of $H_{\mathcal{W}^{M_C}}$, i.e.

$$\mathcal{W}_0^{M_C} = \{ w \in \mathcal{W}^{M_C} \mid H_{\mathcal{W}^{M_C}}(w) = (\hat{C} - \hat{\mathbb{L}})(w) = 0 \} \; , \tag{36}$$

and denote by $\jmath_0^{M_C} \colon \mathcal{W}_0^{M_C} \hookrightarrow \mathcal{W}^{M_C}$ the natural embedding. As in Proposition 1 we may prove the following:

Proposition 6 *$\mathcal{W}_0^{M_C}$ is a 1-codimensional $\mu_{\mathcal{W}^{M_C}}$-transverse submanifold of $\mathcal{W}^{M_C}$, diffeomorphic to $\mathcal{W}_r^{M_C}$.*

As a consequence, the submanifold $\mathcal{W}_0^{\mathcal{F}}$ induces a section of the projection $\mu_{\mathcal{W}^{M_C}}$,

$$\hat{h}^{M_C} \colon \mathcal{W}_r^{M_C} \longrightarrow \mathcal{W}^{M_C} \ .$$

Then we can draw the following diagram, which is analogous to (27)

$$\begin{tikzcd}
 & & C \times_E J^1\pi \arrow[dll] \arrow[drr] & & \\
E & & M_C \arrow[ll, "\pi^{M_C}"'] \arrow[rr, "\bar{\pi}^{M_C}"] \arrow[u, "\jmath^{M_C}"] & & \mathbb{R} \\
\mathcal{W}_0^{M_C} \arrow[u, "\rho_E^{0M_C}"] \arrow[urr, "\rho_1^{0M_C}"] \arrow[rr, "\jmath_0^{M_C}"] \arrow[drr, "\rho_2^{0M_C}"] \arrow[ddrr, "\hat{\rho}_2^{0M_C}"'] & & \mathcal{W}^{M_C} \arrow[ull, "\rho_E^{M_C}"] \arrow[u, "\rho_1^{M_C}"] \arrow[urr, "\rho_{\mathbb{R}}^{M_C}"] \arrow[rr, "\mu_{\mathcal{W}^{M_C}}"] \arrow[d, "\rho_2^{M_C}"] & & \mathcal{W}_r^{M_C} \arrow[ull, "\rho_1^{rM_C}"] \arrow[u, "\rho_{\mathbb{R}}^{rM_C}"'] \arrow[dll, "\rho_2^{rM_C} \circ \hat{h}^{M_C}"'] \arrow[ddll, "\rho_2^{rM_C}"] \\
 & & T^*E \arrow[d, "\mu"] & & \\
 & & J^1\pi^* & &
\end{tikzcd}$$

Finally, we define the forms

$$\Theta_{\mathcal{W}_0^{M_C}} = (\jmath_0^{M_C})^*\Theta_{\mathcal{W}^{M_C}} \quad , \quad \Omega_{\mathcal{W}_0^{M_C}} = (\jmath_0^{M_C})^*\Omega_{\mathcal{W}^{M_C}} \ .$$

4.2 Optimal Control equations

Now, we will see how the dynamics of the optimal control problem $(\mathbf{L}, M_C)$ is determined by the solutions (where they exist) of the equation

$$i(Z)\Omega_{\mathcal{W}_0^{M_C}} = 0, \qquad i(Z)\mathrm{d}t = 1 \quad , \quad \text{for } Z \in \mathfrak{X}(\mathcal{W}_0^{M_C}) \ . \tag{37}$$

As in Section 3.3 , the second equation of (37) can be relaxed to the condition

$$i(Z)\mathrm{d}t \neq 0 \ .$$

In order to work in local coordinates we need the following proposition, whose proof is obvious:

Proposition 7 *$w \in \mathcal{W}_0^{M_C}$ being fixed, the following conditions are equivalent:*

1. There exists a vector $Z_w \in T_w\mathcal{W}_0^{M_C}$ *verifying that*

$$\Omega_{\mathcal{W}_0^{M_C}}(Z_w, Y_w) = 0 \ , \text{ for every } Y_w \in T_w\mathcal{W}_0^{M_C} \ .$$

2. There exists a vector $Z_w \in T_w(C \times_E \mathcal{W})$ *verifying that*

(i) $Z_w \in T_w\mathcal{W}_0^{M_C}$*,*

(ii) $i(Z_w)(\sigma^*_{\mathcal{W}}\Omega_{\mathcal{W}})_w \in (T_w\mathcal{W}_0^{M_C})^0$.

In this last proposition, we use condition 2 to obtain the implicit optimal control equations. Observe that this condition 2 can be understood as follows: there exist $Z \in \mathfrak{X}(C \times_E \mathcal{W})$ such that

(i) Z is tangent to $\mathcal{W}_0^{M_C}$.

(ii) The 1-form $i(Z)\sigma^*_{\mathcal{W}}\Omega_{\mathcal{W}}$ is null on the vector fields tangent to $\mathcal{W}_0^{M_C}$.

As $\mathcal{W}_0^{M_C}$ is defined in (36), and the constraints are $\Psi^\alpha = 0$ and $\hat{C} - \hat{\mathbb{L}} = 0$; then there exist $\lambda_\alpha, \lambda \in \mathrm{C}^\infty(C \times_E \mathcal{W})$, to be determined, such that

$$(i(Z)\sigma^*_{\mathcal{W}}\Omega_{\mathcal{W}})|_{\mathcal{W}_0^{M_C}} = (\lambda_\alpha \mathrm{d}\Psi^\alpha + \lambda \mathrm{d}(\hat{C} - \hat{\mathbb{L}}))|_{\mathcal{W}_0^{M_C}} \ .$$

As usual, the undetermined functions λ_α's and λ are called Lagrange multipliers.

Now using coordinates $(t, q^i, u^a, v^i, p, p^i)$ in $C \times_E \mathcal{W}$, we look for a vector field

$$Z = \frac{\partial}{\partial t} + A^i\frac{\partial}{\partial q^i} + B^a\frac{\partial}{\partial u^a} + C^i\frac{\partial}{\partial v^i} + D_i\frac{\partial}{\partial p_i} + E\frac{\partial}{\partial p} \ ,$$

where A^i, B^a, C^i, D_i, E are unknown functions in $\mathcal{W}_0^{M_C}$ verifying the equation

$$\begin{aligned}
0 &= i_Z\left(\mathrm{d}q^i \wedge \mathrm{d}p_i + \mathrm{d}t \wedge \mathrm{d}p\right) - \lambda_\alpha \mathrm{d}\Psi^\alpha - \lambda \mathrm{d}(p + p_i v^i - \mathbb{L}(q,u,t)) \\
&= \left(-E - \lambda_\alpha\frac{\partial\Psi^\alpha}{\partial t} + \lambda\frac{\partial\mathbb{L}}{\partial t}\right)\mathrm{d}t + \left(\lambda\frac{\partial\mathbb{L}}{\partial q^i} - \lambda_\alpha\frac{\partial\Psi^\alpha}{\partial q^i} - D_i\right)\mathrm{d}q^i \\
&\quad + \left(\lambda\frac{\partial\mathbb{L}}{\partial u^a} - \lambda_\alpha\frac{\partial\Psi^\alpha}{\partial u^a}\right)\mathrm{d}u^a + \left(-\lambda p_i - \lambda_\alpha\frac{\partial\Psi^\alpha}{\partial v^i}\right)\mathrm{d}v^i \\
&\quad + (A^i - \lambda v^i)\mathrm{d}p_i + (1-\lambda)\mathrm{d}p \ .
\end{aligned}$$

Thus, we obtain $\lambda = 1$, and

$$A^i = v^i \ , \ D_i = \frac{\partial\mathbb{L}}{\partial q^i} - \lambda_\alpha\frac{\partial\Psi^\alpha}{\partial q^i} \ , \ E = \frac{\partial\mathbb{L}}{\partial t} - \lambda_\alpha\frac{\partial\Psi^\alpha}{\partial t} \ , \ p_i = -\lambda_\alpha\frac{\partial\Psi^\alpha}{\partial v^i} \ , \ 0 = \frac{\partial\mathbb{L}}{\partial u^a} - \lambda_\alpha\frac{\partial\Psi^\alpha}{\partial u^a}$$

together with the tangency conditions

$$\begin{aligned}
0 &= Z(\Psi^\alpha)|_{\mathcal{W}_0^{M_C}} = \left(\frac{\partial\Psi^\alpha}{\partial t} + A^i\frac{\partial\Psi^\alpha}{\partial q^i} + B^a\frac{\partial\Psi^\alpha}{\partial u^a} + C^i\frac{\partial\Psi^\alpha}{\partial v^i}\right)\Big|_{\mathcal{W}_0^{M_C}} \\
0 &= Z(p + p_i v^i - \mathbb{L}(q,u,t))|_{\mathcal{W}_0^{M_C}} \ .
\end{aligned}$$

Therefore the equations of motion are:

$$\begin{aligned}
\frac{d}{dt}\left(\lambda_\alpha(t)\frac{\partial\Psi^\alpha}{\partial v^i}(t,q(t),\dot{q}(t),u(t))\right) + \frac{\partial\mathbb{L}}{\partial q^i}(t,q(t),\dot{q}(t),u(t)) - \lambda_\alpha(t)\frac{\partial\Psi^\alpha}{\partial q^i}(t,q(t),\dot{q}(t),u(t)) &= 0 \\
\frac{\partial\mathbb{L}}{\partial u^a}(t,q(t),\dot{q}(t),u(t)) - \lambda_\alpha(t)\frac{\partial\Psi^\alpha}{\partial u^a}(t,q(t),\dot{q}(t),u(t)) &= 0 \\
\Psi^\alpha(t,q(t),\dot{q}(t),u(t)) &= 0
\end{aligned}$$

Remark: In the particular case that $\Psi^j = v^j - \mathcal{F}^j = 0$, the vector field Z so-obtained is just the image of the vector field obtained in Section 3.3 by the Hamiltonian section (26), as a simple calculation in coordinates shows.

5 Applications and examples

5.1 Optimal Control of Lagrangian systems with controls

See Appendix A for previous geometric concepts which are needed in this section. For a complete study of these systems see [2, 4] and references therein.

Now we provide a definition of a *controlled-force*, which allows dependence on time, configuration, velocities and control inputs. In a global description, one assumes a fiber bundle structure $\Phi^{1C}\colon C \longrightarrow J^1\pi$, where C is the bundle of controls, with coordinates (t, q, v, u). Then a controlled-force is a smooth map $\mathcal{F}\colon C \longrightarrow \mathcal{C}_\pi$, so that the following diagram commutes.

$$\begin{tikzcd} C \arrow[rr, "\mathcal{F}"] \arrow[dr, "\Phi^{1C}"'] & & \mathcal{C}_\pi \arrow[dl] \\ & J^1\pi & \end{tikzcd}$$

In a natural chart, a controlled-force is represented by

$$\mathcal{F}(t, q, v, u) = \mathcal{F}_i(t, q, v, u)(\mathrm{d}q^i - v^i \mathrm{d}t) \ .$$

A *controlled Lagrangian system* is defined as the pair $(\mathcal{L}, \mathcal{F})$ which determines an implicit control system described by the subset D_C of $C \times_{J^1\pi} J^2\pi$:

$$\begin{aligned} D_C &= \{(c, \hat{p}) \in C \times_{J^1\pi} J^2\pi \mid (\imath_1^* d_T \Theta_{\mathcal{L}} - (\pi_1^2)^* \mathrm{d}L)(\hat{p}) = ((\pi_1^2)^* \mathcal{F})(c)\} \\ &= \{(c, \hat{p}) \in C \times_{J^1\pi} J^2\pi \mid \mathcal{E}_{\mathcal{L}}(\hat{p}) = ((\pi_1^2)^* \mathcal{F})(c)\} \\ &= \{(c, \hat{p}) \in C \times_{J^1\pi} J^2\pi \mid (\mathcal{E}_{\mathcal{L}} \circ pr_2 - (\pi_1^2)^* \mathcal{F} \circ pr_1)(c, \hat{p}) = 0\} \end{aligned}$$

where pr_1 and pr_2 are the natural projections from $C \times_{J^1\pi} J^2\pi$ onto the factors. In fact, D_C is not necessarily a submanifold of $C \times_{J^1\pi} J^2\pi$. There are a lot of cases where this does happen. In local coordinates

$$\begin{aligned} D_C = & \left\{ (t, q, v, w, u) \in J^2\pi \;\middle|\; \frac{\partial^2 L}{\partial v^i \partial v^j}(t, q, v) w^j + \frac{\partial^2 L}{\partial v^i \partial q^j}(t, q, v) v^j \right. \\ & \left. + \frac{\partial^2 L}{\partial v^i \partial t}(t, q, v) - \frac{\partial L}{\partial q^i}(t, q, v) - \mathcal{F}_i(t, q, v, u) = 0 \right\} \ . \end{aligned}$$

A solution to the controlled Lagrangian system $(\mathcal{L}, \mathcal{F})$ is a map $\gamma\colon \mathbb{R} \longrightarrow C$ satisfying that:

(i) $\Phi^{1C} \circ \gamma = j^1(\pi^1 \circ \Phi^{1C} \circ \gamma)$.

(ii) $(\gamma(t), j^2(\pi^1 \circ \Phi^{1C} \circ \gamma)(t)) \in D_C$, for every $t \in \mathbb{R}$.

The condition (i) means that $\Phi^{1C} \circ \gamma$ is holonomic, and (ii) is the condition (47) of Appendix A.3; that is, the Euler-Lagrange equations for the controlled Lagrangian system $(\mathcal{L}, \mathcal{F})$.

Now, consider the map $(\mathrm{Id},\Upsilon)\colon C\times_{J^1\pi} J^2\pi \longrightarrow C\times_{J^1\pi} J^1\bar{\pi}^1$, where $\Upsilon\colon J^2\pi\longrightarrow J^1\bar{\pi}^1$ is defined in (46) (see Appendix A.2), and let $M_C=(\mathrm{Id},\Upsilon)(D_C)$. As (Id,Υ) is an injective map, we can identify $D_C\subset C\times_{J^1\pi}J^2\pi$ with this subset M_C of $C\times_{J^1\pi}J^1\bar{\pi}^1$. Observe that there is a natural projection from M_C to $J^1\pi$.

If $\mathbb{L}\colon M_C\longrightarrow\mathbb{R}$ is a cost function, we may consider the implicit optimal control system determined by the pair $(\mathbf{L},M_C)$, where $\mathbf{L}=\mathbb{L}\mathrm{d}t$, and apply the method developed in Section 4.

Let $\overline{\mathcal{W}}^{M_C}=M_C\times_{J^1\pi}T^*J^1\pi$, and $\overline{\mathcal{W}}^C=C\times_{J^1\pi}J^1\bar{\pi}^1\times_{J^1\pi}T^*J^1\pi$. The natural projection from $\overline{\mathcal{W}}^C$ to $T^*J^1\pi$ allows us to pull-back the canonical 2-form $\Omega_{J^1\pi}$ to a presymplectic form $\Omega_{\overline{\mathcal{W}}^C}\in\Omega^2(\overline{\mathcal{W}}^C)$. Furthermore, in $J^1\bar{\pi}^1\times_{J^1\pi}T^*J^1\pi$ there is the natural coupling form $\bar{\hat{\mathcal{C}}}$ (see Definition 1). We denote by $\bar{\mathcal{C}}$ its pull-back to $\overline{\mathcal{W}}^C$. We denote by $\mathbf{L}$ and $\mathbb{L}$ the pull-back of $\mathbf{L}$ and $\mathbb{L}$ from M_C to $\overline{\mathcal{W}}^C$, for the sake of simplicity.

Then, let $\bar{H}_{\mathcal{W}^C}\colon\overline{\mathcal{W}}^C\longrightarrow\mathbb{R}$ be the unique function such that $\bar{\mathcal{C}}-\mathbf{L}=\bar{H}_{\mathcal{W}^C}\mathrm{d}t$, whose local expression is $\bar{H}_{\mathcal{W}^C}=p+p_i\bar{v}^i+\bar{p}_iw^i-\mathbb{L}$, and consider the submanifold $\overline{\mathcal{W}}_0=\{\tilde{q}\in\overline{\mathcal{W}}^C\mid\bar{H}_{\mathcal{W}^C}(\tilde{q})=0\}$. The pull-back of $\bar{H}_{\mathcal{W}^C}$ to $\overline{\mathcal{W}}^{M_C}$ is the Pontryaguin Hamiltonian, denoted by $\bar{H}_{\mathcal{W}^{M_C}}$.

Finally, the dynamics is in the submanifold $\overline{\mathcal{W}}_0^{M_C}=\overline{\mathcal{W}}^{M_C}\cap\overline{\mathcal{W}}_0$ of $\overline{\mathcal{W}}^C$, where $\jmath_1^{M_C}$ is the natural embedding. $\overline{\mathcal{W}}_0^{M_C}$ is endowed with the presymplectic form $\Omega_{\overline{\mathcal{W}}_0^{M_C}}=(\jmath_1^{M_C})^*\Omega_{\overline{\mathcal{W}}^C}$. Therefore, the motion is determined by a vector field $Z\in\mathfrak{X}(\overline{\mathcal{W}}_0^{M_C})$ satisfying the equations

$$i(Z)\Omega_{\overline{\mathcal{W}}_0^{M_C}}=0\quad,\quad i(Z)\mathrm{d}t=1\ .$$

A local chart in $\overline{\mathcal{W}}^C$ is $(t,q^i,v^i,\bar{v}^i,w^i,u^\alpha,p,p_i,\bar{p}_i)$, where $(\bar{v}^i,w^i)$ and $(p,p_i,\bar{p}_i)$ are the natural fiber coordinates in $J^1\bar{\pi}^1$ and $\mathrm{T}^*J^1\pi$, respectively. The manifold $\overline{\mathcal{W}}^{M_C}$ is given locally by the $2n$ constraints:

$$\begin{aligned}
\varphi_i(t,q^i,v^i,\bar{v}^i,w^i,u^\alpha,p,p_i,\bar{p}_i) &= w^j\frac{\partial^2L}{\partial v^i\partial v^j}(t,q,v)+\bar{v}^j\frac{\partial^2L}{\partial v^i\partial q^j}(t,q,v)+\frac{\partial^2L}{\partial v^i\partial t}(t,q,v)\\
&\quad-\frac{\partial L}{\partial q^i}(t,q,v)-\mathcal{F}_i(t,q,v,u)=0\\
\bar{\varphi}^i(t,q^i,v^i,\bar{v}^i,w^i,u^\alpha,p,p_i,\bar{p}_i) &= v^i-\bar{v}^i=0\ ,
\end{aligned}$$

and $\overline{\mathcal{W}}_0$ is given by

$$\phi(t,q^i,v^i,\bar{v}^i,w^i,u^\alpha,p,p_i,\bar{p}_i)=\bar{H}_{\mathcal{W}^C}(t,q^i,v^i,\bar{v}^i,w^i,u^\alpha,p,p_i,\bar{p}_i)=p+p_i\bar{v}^i+\bar{p}_iw^i-\mathbb{L}(t,q,v,u)=0\ ,$$

and

$$\Omega_{\overline{\mathcal{W}}_0^{M_C}}=\mathrm{d}q^i\wedge\mathrm{d}p_i+\mathrm{d}v^i\wedge\mathrm{d}\bar{p}_i+\mathrm{d}t\wedge\mathrm{d}(\mathbb{L}-p_i\bar{v}^i-\bar{p}_iw^i)\ .$$

Following Proposition 7, we look for a vector field $Z\in\mathfrak{X}(\overline{\mathcal{W}}^C)$ such that, for every $\mathbf{w}\in\overline{\mathcal{W}}_0^{M_C}$:

$$\text{(i)}\quad Z_{\mathbf{w}}\in\mathrm{T}_{\mathbf{w}}\overline{\mathcal{W}}_0^{M_C}\quad,\quad\text{(ii)}\quad i(Z_{\mathbf{w}})\Omega_{\overline{\mathcal{W}}^C}\in(\mathrm{T}_{\mathbf{w}}\overline{\mathcal{W}}_0^{M_C})^0\ ,$$

or, equivalently

(i) $(\jmath_1^{M_C})^*(Z(\varphi_i))=0$, $\quad(\jmath_1^{M_C})^*(Z(\bar{\varphi}_i))=0$, $\quad(\jmath_1^{M_C})^*(Z(\phi))=0$.

(ii) $(\jmath_1^{M_C})^*(i(Z)\Omega_{\overline{\mathcal{W}}^C})=0$.

Remember that the constraints are $\varphi_i = 0, \quad \bar{\varphi}_i = 0, \quad \phi = 0$.

If Z is given locally by

$$Z = \frac{\partial}{\partial t} + A^i \frac{\partial}{\partial q^i} + \mathcal{A}^i \frac{\partial}{\partial v^i} + \bar{A}^i \frac{\partial}{\partial \bar{v}^i} + \bar{\mathcal{A}}^i \frac{\partial}{\partial w^i} + B^a \frac{\partial}{\partial u^a} + D \frac{\partial}{\partial p} + C_i \frac{\partial}{\partial p_i} + \bar{C}_i \frac{\partial}{\partial \bar{p}_i} ,$$

then $A^i, \mathcal{A}^i, \bar{A}^i, \bar{\mathcal{A}}^i, B^a, D, C_i, \bar{C}_i$ are unknown functions in $\overline{\mathcal{W}}^C$, such that

$$i(Z)\Omega_{\overline{\mathcal{W}}^C} = \lambda^i \mathrm{d}\varphi_i + \bar{\lambda}_i \mathrm{d}\bar{\varphi}^i + \lambda \mathrm{d}(p + p_i \bar{v}^i + \bar{p}_i w^i - \mathbb{L}(t, q, v, u))$$

and $Z(\varphi_i) = 0$, $Z(\bar{\varphi}^i) = 0$ and $Z(p + p_i \bar{v}^i + \bar{p}_i w^i - \mathbb{L}(t, q, v, u)) = 0$. From these equations we obtain

$$\begin{array}{ccc} \lambda = 1 \, , & A^i = \bar{v}^i \, , & \mathcal{A}^i = w^i \\ C_i = \dfrac{\partial \mathbb{L}}{\partial q^i} - \lambda^j \dfrac{\partial \varphi_j}{\partial q^i} \, , & \bar{C}_i = \dfrac{\partial \mathbb{L}}{\partial v^i} - \lambda^j \dfrac{\partial \varphi_j}{\partial v^i} - \bar{\lambda}_i \, , & D = \dfrac{\partial \mathbb{L}}{\partial t} - \lambda^j \dfrac{\partial \varphi_j}{\partial t} \\ 0 = \dfrac{\partial \mathbb{L}}{\partial u^a} + \lambda^i \dfrac{\partial \mathcal{F}_i}{\partial u^a} \, , & p_i = \bar{\lambda}_i - \lambda^j \dfrac{\partial^2 L}{\partial v^j \partial q^i} \, , & \bar{p}_i = -\lambda^j \dfrac{\partial^2 L}{\partial v^i \partial v^j} \end{array} \tag{38}$$

and the tangency conditions

$$\begin{aligned} Z(\varphi_i) &= \frac{\partial \varphi_i}{\partial t} + \bar{v}^j \frac{\partial \varphi_i}{\partial q^j} + w^j \frac{\partial \varphi_i}{\partial v^j} + \bar{A}^j \frac{\partial^2 L}{\partial v^i \partial q^j} - B^a \frac{\partial \mathcal{F}_i}{\partial u^a} + \bar{\mathcal{A}}^j \frac{\partial^2 L}{\partial v^i \partial v^j} = 0 \qquad (39) \\ Z(\bar{\varphi}^i) &= w^i - \bar{A}^i = 0 \\ Z(\phi) &= Z(p + p_i \bar{v}^i + \bar{p}_i w^i - \mathbb{L}(t, q, v, u)) = 0 \end{aligned}$$

where the third condition is satisfied identically using the previous equations.

Assuming that the Lagrangian L is regular, that is, $\det(W_{ij}) = det\left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right) \neq 0$, then from equations for p_i and $\bar{p}_i$ in (38) we obtain explicit values of the Lagrange multipliers λ^i and $\bar{\lambda}_i$. Therefore, the remaining equations (38) are now rewritten as the new set of constraints

$$\psi^a(t, q, v, u, \bar{p}) = \frac{\partial \mathbb{L}}{\partial u^a} - W^{ij} \bar{p}_i \frac{\partial \mathcal{F}_j}{\partial u^a} = 0 \, , \tag{40}$$

which corresponds to $\dfrac{\partial \bar{H}_{\mathcal{W}^{MC}}}{\partial u^a} = 0$.

The new compatibility condition is

$$Z(\psi^a) = \frac{\partial \psi^a}{\partial t} + \bar{v}^j \frac{\partial \psi^a}{\partial q^j} + w^j \frac{\partial \psi^a}{\partial v^j} + B^b \frac{\partial \psi^a}{\partial u^b} + \bar{C}_i \frac{\partial \psi^a}{\partial \bar{p}_i} = 0 \, . \tag{41}$$

Furthermore we assume that

$$\det\left(\frac{\partial \psi^a}{\partial u^b}\right) \neq 0 \, ,$$

then, from Equations (39) and (41) we obtain the remaining components $\bar{\mathcal{A}}^i$ and B^a, and we determine completely the vector field Z.

The equations of motion for a curve are determined by the system of implicit-differential equations:

$$\begin{aligned}
\dot{p}_i(t) &= \frac{\partial \mathbb{L}}{\partial q^i}(t,q(t),\dot{q}(t),u(t)) - \lambda^j(t,q(t),\dot{q}(t),\bar{p}(t))\frac{\partial \varphi_j}{\partial q^i}(t,q(t),\dot{q}(t),\ddot{q}(t),u(t)) \\
\dot{\bar{p}}_i(t) &= \frac{\partial \mathbb{L}}{\partial v^i}(t,q(t),\dot{q}(t),u(t)) - p_i(t) \\
&\quad -\lambda^j(t,q(t),\dot{q}(t),\bar{p}(t))\left[\frac{\partial \varphi_j}{\partial v^i}(t,q(t),\dot{q}(t),\ddot{q}(t),u(t)) + \frac{\partial^2 L}{\partial v^j \partial q^i}(t,q(t),\dot{q}(t))\right] \quad (42)\\
0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(t,q(t),\dot{q}(t))\right) - \frac{\partial L}{\partial q^i}(t,q(t),\dot{q}(t)) - \mathcal{F}_i(t,q(t),\dot{q}(t),u(t)) \quad (43)\\
0 &= \frac{\partial \mathbb{L}}{\partial u^a}(t,q(t),\dot{q}(t),u(t)) - W^{ij}(t,q(t),\dot{q}(t))\bar{p}_i(t)\frac{\partial \mathcal{F}_j}{\partial u^a}(t,q(t),\dot{q}(t),u(t)) \ . \quad (44)
\end{aligned}$$

Equation (44) is the explicit expression of (40).

In [1] the authors study optimal control of Lagrangian systems with controls in a more restrictive situation using higher-order dynamics, obtaining that the states are determined by a set of fourth-order differential equations. First it is necessary to assume that the system is *fully actuated*, that is $m = n$, and $\operatorname{rank}(\Xi_{ij}) = \operatorname{rank}\left(\frac{\partial \mathcal{F}_i}{\partial u^j}\right) = n$. Moreover, in the sequel we assume that the system is affine on controls, that is,

$$\mathcal{F}_i(t,q,\dot{q},u) = A_i(t,q,\dot{q}) + A_{ij}(t,q,\dot{q})u^j \ .$$

Therefore, $\Xi_{ij} = A_{ij}$.

Then from the constraint equations (43) and (44), applying the Implicit Function Theorem, we deduce that

$$\begin{aligned}
u^i(t) &= u^i(t,q(t),\dot{q}(t),\ddot{q}(t)) = A^{ij}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial v^j}(t,q(t),\dot{q}(t))\right) - \frac{\partial L}{\partial q^j}(t,q(t),\dot{q}(t)) - A_j(t,q(t),\dot{q}(t))\right] \\
\bar{p}_i(t) &= \mathcal{H}_i^j(t,q(t),\dot{q}(t))\frac{\partial \mathbb{L}}{\partial u^j}(t,q(t),\dot{q}(t),u(t,q(t),\dot{q}(t),\ddot{q}(t)))
\end{aligned}$$

where $(\mathcal{H}_i^j)$ are the components of the inverse matrix of the regular matrix $(W^{ik}A_{kj})$.

Taking the derivative with respect to time of Equation (42), and substituting the value of $\dot{p}_i(t)$ using Equation (42) we obtain a fourth-order differential equation depending on the states. After some computations we deduce that

$$\mathcal{H}_i^j(t,q(t),\dot{q}(t))\frac{\partial^2 \mathbb{L}}{\partial u^j \partial u^k}(t,q(t),\dot{q}(t),\ddot{q}(t))\frac{d^4 q^k}{dt^4}(t) = G_i(t,q(t),\dot{q}(t),\ddot{q}(t),\dddot{q}(t)) \ .$$

Finally, under the assumption that the matrix $\left(\frac{\partial^2 \mathbb{L}}{\partial u^j \partial u^k}\right)$ is invertible, we obtain a explicit fourth-order system of differential equations:

$$\frac{d^4 q^i}{dt^4}(t) = \bar{G}^i(t,q(t),\dot{q}(t),\ddot{q}(t),\dddot{q}(t)) \ .$$

5.2 Optimal Control problems for descriptor systems

See [17] for the origin and interest of this example. The study of these kinds of systems was suggested to us by Professor. A.D. Lewis (Queen's University of Canada).

Consider the problem of minimizing the functional

$$\mathcal{J} = \frac{1}{2}\int_0^{+\infty}\left[a_i(q^i)^2 + ru^2\right]\,\mathrm{d}t,$$

$1 \leq i \leq 3$, with control equations

$$\dot{q}^2 = q^1 + b_1 u \quad , \quad \dot{q}^3 = q^2 + b_2 u \quad , \quad 0 = q^3 + b_3 u$$

with parameters $a_i, b_i \geq 0$ and $r > 0$.

As in the previous section, the geometric framework developed in Section 3.2 is also valid for this class of systems. Let $E = \mathbb{R} \times \mathbb{R}^3$ with coordinates (t, q^i), and $C = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$ with coordinates (t, q^i, u). The submanifold $M_C \subset C \times_E J^1\pi$ is given by

$$M_C = \{(t, q^1, q^2, q^3, v^1, v^2, v^3, u) \mid v^2 = q^1 + b_1 u \,, v^3 = q^2 + b_2 u \,, 0 = q^3 + b_3 u\} \,.$$

The cost function is

$$\begin{array}{rccl} \mathbb{L} : & C & \longrightarrow & \mathbb{R} \\ & (t, q^1, q^2, q^3, u) & \longmapsto & \dfrac{1}{2}\left[a_1 (q^1)^2 + a_2 (q^2)^2 + a_3 (q^3)^2 + r u^2\right] \end{array}$$

We analyze the dynamics of the implicit optimal control system determined by the pair $(\mathbf{L}, M_C)$.

Let $\mathcal{W}^{M_C} = M_C \times_E T^*E$ and $\mathcal{W}^C = C \times_E J^1\pi \times_E T^*E$ with coupling form $\mathcal{C}$ inherited from the natural coupling form in $J^1\pi \times T^*E$. Let $H_{\mathcal{W}^C} \colon \mathcal{W}^C \longrightarrow \mathbb{R}$ be the unique function such that $\mathcal{C} - \mathbf{L} = H_{\mathcal{W}^C} \mathrm{d}t$, and consider the submanifold $\mathcal{W}_0 = \{\tilde{q} \in \mathcal{W}^C \mid H_{\mathcal{W}^C}(\tilde{q}) = 0\}$. Finally, the dynamics is in the submanifold $\mathcal{W}_0^{M_C} = \mathcal{W}^{M_C} \cap \mathcal{W}_0$ of $\mathcal{W}^C$. Locally,

$$\begin{aligned} \mathcal{W}_0^{M_C} = \; & \{(t, q^1, q^2, q^3, v^1, v^2, v^3, u, p, p_1, p_2, p_3) \mid v^2 = q^1 + b_1 u \,, v^3 = q^2 + b_2 u \,, \\ & q^3 + b_3 u = 0 \,, p + p_1 v^1 + p_2 v^2 + p_3 v^3 - \mathbb{L} = 0\} \,. \end{aligned}$$

Therefore, the motion is determined by a vector field $Z \in \mathfrak{X}(\mathcal{W}_0^{M_C})$ satisfying the Equations (37), which according to Proposition 7 is equivalent to finding a vector field $Z \in \mathfrak{X}(\mathcal{W}^C)$ (if it exists):

$$Z = \frac{\partial}{\partial t} + A^1 \frac{\partial}{\partial q^1} + A^2 \frac{\partial}{\partial q^2} + A^3 \frac{\partial}{\partial q^3} + C^1 \frac{\partial}{\partial v^1} + C^2 \frac{\partial}{\partial v^2} + C^3 \frac{\partial}{\partial v^3} + B \frac{\partial}{\partial u} + D_1 \frac{\partial}{\partial p_1} + D_2 \frac{\partial}{\partial p_2} + D_3 \frac{\partial}{\partial p_3} + E \frac{\partial}{\partial p}$$

such that

$$\begin{array}{c} i(Z)\Omega_{\mathcal{W}^C} = \lambda_1 d(q^1 + b_1 u - v^2) + \lambda_2 d(q^2 + b_2 u - v^3) + \lambda_3 d(q^3 + b_3 u) + \lambda d H_{\mathcal{W}^C} \,, \\ Z(q^1 + b_1 u - v^2) = 0 \,, \quad Z(q^2 + b_2 u - v^3) = 0 \,, \quad Z(q^3 + b_3 u) = 0 \,, \quad Z(H_{\mathcal{W}^C}) = 0 \end{array}$$

where $\Omega_{\mathcal{W}^C} \in \Omega^2(\mathcal{W}^C)$ is the 2-form with local expression

$$\Omega_{\mathcal{W}^C} = \mathrm{d}q^1 \wedge \mathrm{d}p_1 + \mathrm{d}q^2 \wedge \mathrm{d}p_2 + \mathrm{d}q^3 \wedge \mathrm{d}p_3 + \mathrm{d}t \wedge \mathrm{d}p \,.$$

After some straightforward computations, we obtain that

$$\begin{array}{rcccl} A^1 = v^1 & , & A^2 = q^1 + b_1 u & , & A^3 = q^2 + b_2 u \\ \lambda = 1 & , & E = 0 & , & 0 = r u - b_1 p_2 - b_2 p_3 - b_3 \lambda_3 \\ C^2 = v^1 + b_1 B & , & C^3 = A^2 + b_2 B & , & 0 = A^3 + b_3 B \\ p_1 = 0 & , & p_2 = \lambda_1 & , & p_3 = \lambda_2 \\ D_1 = a_1 q_1 - p_2 & , & D_2 = a_2 q_2 - p_3 & , & D_3 = a_3 q_3 - \lambda_3 \,. \end{array}$$

We deduce that

$$\lambda_3 = \frac{1}{b_3}(r u - b_1 p_2 - b_2 p_3) \,, \quad B = -\frac{1}{b_3}(q^2 + b_2 u) \,.$$

Therefore, the new constraint submanifold $\mathcal{W}_1^{M_C} \hookrightarrow \mathcal{W}_0^{M_C}$ is

$$\mathcal{W}_1^{M_C} = \{(t, q^1, q^2, v^1, u, p_1, p_2, p_3) \mid p_1 = 0\}\,.$$

Consistency of the dynamics implies that

$$0 = Z(p_1) = D_1 = a_1 q_1 - p_2\,.$$

Thus,

$$\mathcal{W}_2^{M_C} = \{(t, q^1, q^2, v^1, u, p_2, p_3) \mid a_1 q_1 - p_2 = 0\}$$

and once again we impose the tangency to the new constraints:

$$0 = Z(a_1 q_1 - p_2) = a_1 v^1 - a_2 q_2 + p_3$$

which implies that

$$\mathcal{W}_3^{M_C} = \{(t, q^1, q^2, v^1, u, p_3) \mid a_1 v^1 - a_2 q^2 + p_3 = 0\}\,.$$

From the compatibility condition

$$0 = Z(a_1 v^1 - a_2 q^2 + p_3)$$

and the constraints we determine the remaining component C^1 of Z:

$$C^1 = \frac{1}{a_1 b_3}\left[(a_2 b_3 - a_1 b_1) q^1 - b_2 a_2 q^2 + (a_2 b_1 b_3 + a_3 b_3^2 + r) u + b_2 a_1 v^1\right]\,.$$

Therefore the equations of motion of the optimal control problem are:

$$\begin{aligned}
\ddot{q}^1(t) &= \frac{1}{a_1 b_3}\left[(a_2 b_3 - a_1 b_1) q^1(t) - a_2 b_2 q^2(t) + (a_2 b_1 b_3 + a_3 b_3^2 + r) u(t) + a_1 b_2 \dot{q}^1(t)\right] \qquad (45)\\
\dot{q}^2(t) &= q^1(t) + b_1 u(t)\\
0 &= q^2(t) + b_2 u(t) - b_3 \dot{u}(t)\,.
\end{aligned}$$

From (45) we deduce that

$$u(t) = \frac{1}{a_2 b_1 b_3 + a_3 b_3^2 + r}\left[(a_1 b_1 - a_2 b_3) q^1(t) + a_2 b_2 q^2(t) - a_1 b_2 \dot{q}^1(t) + a_1 b_3 \ddot{q}^1(t)\right]\,.$$

This is the result obtained in Müller [17], where the optimal feedback control depends on the state variables and also on their derivatives (non-casuality).

Choosing local coordinates (t, q^1, q^2, v^1, u) on $\mathcal{W}_3^{M_C}$, if $\jmath_3 : \mathcal{W}_3^{M_C} \mapsto \mathcal{W}^C$ is the canonical embedding, then $\Omega_{\mathcal{W}_3^{M_C}} = \jmath_3^* \Omega_{\mathcal{W}^C}$ is locally written as

$$\Omega_{\mathcal{W}_3^{M_C}} = -a_1 \mathrm{d}q^1 \wedge \mathrm{d}q^2 + a_2 b_3 \mathrm{d}q^2 \wedge \mathrm{d}u - a_1 b_3 \mathrm{d}v^1 \wedge \mathrm{d}u + \mathrm{d}t \wedge \mathrm{d}\jmath_3^* p\,,$$

where $\jmath_3^* p : \mathcal{W}_3^{M_C} \longrightarrow \mathbb{R}$ is the function

$$\jmath_3^* p = -\frac{1}{2} a_1 (q^1)^2 - \frac{1}{2} a_2 (q^2)^2 + \frac{1}{2}(r + a_3 b_3^2) u^2 - a_1 b_1 q^1 u - a_2 b_2 q^2 u + a_1 b_2 v^1 u + a_1 q^2 v^1\,.$$

Obviously, $(\Omega_{\mathcal{W}_3^{M_C}}, \mathrm{d}t)$ is a cosymplectic structure on $\mathcal{W}_3^{M_C}$, and there exists a unique vector field $\bar{Z} \in \mathfrak{X}(\mathcal{W}_3^{M_C})$ satisfying

$$i(\bar{Z})\Omega_{\mathcal{W}_3^{M_C}} = 0, \quad i(\bar{Z})\mathrm{d}t = 1\,.$$

6 Conclusions and outlook

In this paper we have elucidated the geometrical structure of optimal control problems using a variation of the Skinner-Rusk formalism for mechanical systems. The geometric framework allows us to find the dynamical equations of the problem (equivalent to the Pontryaguin Maximum Principle for smooth enough problems without boundaries on the space of controls), and to describe the submanifold (if it exists) where the solutions of the problem are consistently defined. The method admits a nice extension for studying the dynamics of implicit optimal control problems with a wide range of applicability.

One line of future research appears when we combine our geometric method for optimal control problems, and the study of the (approximate) solutions to optimal control problems involving partial differential equations when we discretize the space domain and consider the resultant set of ordinary differential equations (see, for instance, [5] and references therein and [14], for a geometrical description). This resultant system is an optimal control problem, where the state equations are, presumably, a very large set of coupled ordinary differential equations. Typically, difficulties other than computational ones appear because the system is differential-algebraic, and therefore the optimal control problem is a usual one for a descriptor system.

Moreover, in this paper we have confined ourselves to the geometrical aspects of time-dependent optimal control problems. Of course, the techniques are suitable for studying the formalism for optimal control problems for partial differential equations in general.

A Appendix

A.1 Tulczyjew's operators and Euler–Lagrange equations

Given a differentiable manifold Q and its tangent bundle $\tau_Q\colon \mathrm{T}Q \longrightarrow Q$, we consider the following operators, introduced by Tulczyjew [24]: first we introduce $i_T\colon \Omega^k(Q) \longrightarrow \Omega^{k-1}(\mathrm{T}Q)$, which is defined as follows: for every $(\mathrm{p}, v) \in \mathrm{T}Q$, $\alpha \in \Omega^k(Q)$, and $X_1, \ldots, X_{k-1} \in \mathfrak{X}(\mathrm{T}Q)$, we have

$$(i_T\,\alpha)((\mathrm{p}, v); X_1, \ldots, X_{k-1}) = \alpha(\mathrm{p}; v, \mathrm{T}_{(\mathrm{p},v)}\tau_Q((X_1)_{(\mathrm{p},v)}), \ldots, \mathrm{T}_{(\mathrm{p},v)}\tau_Q((X_{k-1})_{(\mathrm{p},v)}))\ .$$

Then, the so-called *total derivative* is a map $\mathrm{d}_T\colon \Omega^k(Q) \longrightarrow \Omega^k(\mathrm{T}Q)$ defined by

$$d_T = \mathrm{d} \circ i_T + i_T \circ \mathrm{d}\ .$$

For the case $k = 1$, using natural coordinates in $\mathrm{T}Q$, the local expression is

$$d_T\alpha \equiv \mathrm{d}_T(A_j \mathrm{d}q^j) = A_j \mathrm{d}v^j + v^i \frac{\partial A_j}{\partial q^i} \mathrm{d}q^j\ .$$

A.2 Some geometrical structures

Recall that, associated with every jet bundle $J^1\pi$, we have the *contact system*, which is a subbundle $\mathcal{C}_\pi$ of $T^*J^1\pi$ whose fibres at every $j^1\phi(t) \in J^1\pi$ are defined as

$$\mathcal{C}_\pi(j^1\phi(t)) = \{\alpha \in \mathrm{T}^*_{j^1\phi(t)}(J^1\pi) \mid \alpha = (\mathrm{T}_{j^1\phi(t)}\pi^1 - \mathrm{T}_{j^1\phi(t)}(\phi \circ \bar{\pi}^1))^*\beta,\ \beta \in \mathrm{V}^*_{\phi(t)}\pi\}\ .$$

One may readily see that a local basis for the sections of this bundle is given by $\{\mathrm{d}q^i - v^i \mathrm{d}t\}$.

Now, denote by $J^2\pi$ the bundle of 2-jets of π. This jet bundle is equipped with natural coordinates (t, q^i, v^i, w^i) and canonical projections

$$\pi^2_1 \colon J^2\pi \longrightarrow J^1\pi, \quad \pi^2 \colon J^2\pi \longrightarrow E \quad , \quad \bar{\pi}^2 \colon J^2\pi \longrightarrow \mathbb{R} \, .$$

Considering the bundle $J^1\bar{\pi}^1$, we introduce the canonical injection $\Upsilon \colon J^2\pi \longrightarrow J^1\bar{\pi}^1$ given by

$$\Upsilon(j^2\phi(t)) = (j^1(j^1\phi))(t) \, . \tag{46}$$

Taking coordinates $(t, q^i, v^i; \bar{v}^i, w^i)$ in $J^1\bar{\pi}^1$ then

$$\Upsilon(t, q^i, v^i, w^i) = (t, q^i, v^i; v^i, w^i) \, .$$

Thus, we have the following diagram

$$\begin{tikzcd}
\mathrm{T}J^1\pi = \mathrm{T}\mathbb{R} \times \mathrm{T}(\mathrm{T}Q) \arrow[ddr, "\tau_{J^1\pi}"'] & T^*(J^2\pi) \arrow[d, "\pi_{J^2\pi}"] & J^1\bar{\pi}^1 = \mathbb{R} \times \mathrm{T}(\mathrm{T}Q) \arrow[ddl, "(\bar{\pi}^1)^1"] \\
 & J^2\pi = \mathbb{R} \times \mathrm{T}^2Q \arrow[ul, "\imath_1"'] \arrow[ur, "\Upsilon"] \arrow[d, "\pi^2_1"] & \\
 & J^1\pi = \mathbb{R} \times \mathrm{T}Q \arrow[ddl, "\pi^1"'] \arrow[ddr, "\bar{\pi}^1"] & \\
 & \mathcal{C}_\pi \subset \mathrm{T}^*J^1\pi \arrow[u, "\pi_{J^1\pi}"] & \\
\mathbb{R} \times Q \arrow[rr, "\pi"] & & \mathbb{R}
\end{tikzcd}$$

where the inclusion $\imath_1$ is locally given by $\imath_1(t, q, v, w) = (t, 1, q, v, v, w)$.

Observe that $(\pi^2_1)^*\mathrm{T}^*J^1\pi$ can be identified with a subbundle of $\mathrm{T}^*J^2\pi$ by means of the natural injection $\hat{\imath} \colon (\pi^2_1)^*\mathrm{T}^*J^1\pi \longrightarrow \mathrm{T}^*J^2\pi$, defined as follows: for every $\hat{p} \in J^2\pi$, $\alpha \in \mathrm{T}^*_{\pi^2_1(\hat{p})}J^1\pi$, and $\mathrm{a} \in \mathrm{T}_{\hat{p}}J^2\pi$,

$$(\hat{\imath}(\hat{p}, \alpha))(\mathrm{a}) = \alpha(\mathrm{T}_{\hat{p}}\pi^2_1(\mathrm{a})) \, .$$

In the same way, we can identify $(\pi^2_1)^*\mathcal{C}_\pi$ as a subbundle of $(\pi^2_1)^*\mathrm{T}^*J^1\pi$ by means of $\hat{\imath}$.

Local bases for the set of sections of the bundles $\mathrm{T}^*J^2\pi \longrightarrow J^2\pi$, $(\pi^2_1)^*\mathrm{T}^*J^1\pi \longrightarrow J^2\pi$, and $(\pi^2_1)^*\mathcal{C}_\pi \longrightarrow J^2\pi$ are $(\mathrm{d}t, \mathrm{d}q^i, \mathrm{d}v^i, \mathrm{d}w^i)$, $(\mathrm{d}t, \mathrm{d}q^i, \mathrm{d}v^i)$, and $(\mathrm{d}q^i - v^i\mathrm{d}t)$, respectively.

Incidentally, $Sec\,(J^2\pi, (\pi^2_1)^*\mathrm{T}^*J^1\pi) = \mathrm{C}^\infty(J^2\pi) \otimes_{\mathrm{C}^\infty(J^1\pi)} (\pi^2_1)^*\mathit{\Omega}^1(J^1\pi)$, which are the π^2_1-semibasic 1-forms in $J^2\pi$.

A.3 Euler-Lagrange equations

Let $\mathcal{L} \in \mathit{\Omega}^1(J^1\pi)$ be a Lagrangian density and its associated Lagrangian function $L \in \mathrm{C}^\infty(J^1\pi)$. Observe that

$$d_T\Theta_{\mathcal{L}} \in \mathit{\Omega}^1(\mathrm{T}J^1\pi) \, , \; \imath_1^* d_T\Theta_{\mathcal{L}} \in \mathit{\Omega}^1(J^2\pi) \, , \; (\pi^2_1)^*\mathrm{d}L \in \mathit{\Omega}^1(J^2\pi) \, .$$

Then, a simple calculation in coordinates shows that $\imath_1^* d_T\Theta_{\mathcal{L}} - (\pi^2_1)^*\mathrm{d}L$ is a section of the bundle projection $\hat{\imath}((\pi^2_1)^*\mathcal{C}_\pi) \longrightarrow J^2\pi$.

The Euler-Lagrange equations for this Lagrangian are a system of second order differential equations on Q; that is, in implicit form, a submanifold D of $J^2\pi$ determined by:

$$D = \{\hat{p} \in J^2\pi \mid (\imath_1^* d_T\Theta_{\mathcal{L}} - (\pi_1^2)^*\mathrm{d}L)(\hat{p}) = 0\} = \{\hat{p} \in J^2\pi \mid \mathcal{E}_{\mathcal{L}}(\hat{p}) = 0\} = \mathcal{E}_{\mathcal{L}}^{-1}(0) \ ,$$

where $\mathcal{E}_{\mathcal{L}} = \imath_1^* d_T\Theta_{\mathcal{L}} - (\pi_1^2)^*\mathrm{d}L$. Then, a section $\phi\colon \mathbb{R} \longrightarrow \mathbb{R}\times Q$ is a solution to the Lagrangian system if, and only if, $\mathrm{Im}\, j^2\phi \subset \mathcal{E}_{\mathcal{L}}^{-1}(0)$. In fact, working in local coordinates, such as

$$\begin{aligned}
d_T\Theta_{\mathcal{L}} &= \frac{\partial L}{\partial v^k}\mathrm{d}v^k - \left(\frac{\partial L}{\partial v^j}v^j - L\right)\mathrm{d}\dot{t} + \left(\dot{t}\frac{\partial^2 L}{\partial t\partial v^k} + v^i\frac{\partial^2 L}{\partial q^i\partial v^k} + w^i\frac{\partial^2 L}{\partial v^i\partial v^k}\right)\mathrm{d}q^k \\
&\quad - \left[\dot{t}\left(v^j\dot{t}\frac{\partial^2 L}{\partial t\partial v^j} - \frac{\partial L}{\partial t}\right) + v^i\left(v^j\frac{\partial^2 L}{\partial q^i\partial v^j} - \frac{\partial L}{\partial q^i}\right) + w^i\left(\frac{\partial L}{\partial v^i} + v^j\frac{\partial^2 L}{\partial v^i\partial v^j} - \frac{\partial L}{\partial v^i}\right)\right]\mathrm{d}t \\
\imath_1^* d_T\Theta_{\mathcal{L}} &= \frac{\partial L}{\partial v^k}\mathrm{d}v^k + \left(\frac{\partial^2 L}{\partial t\partial v^k} + v^i\frac{\partial^2 L}{\partial q^i\partial v^k} + w^i\frac{\partial^2 L}{\partial v^i\partial v^k}\right)\mathrm{d}q^k \\
&\quad - \left[v^j\frac{\partial^2 L}{\partial t\partial v^j} - \frac{\partial L}{\partial t} + v^i\left(v^j\frac{\partial^2 L}{\partial q^i\partial v^j} - \frac{\partial L}{\partial q^i}\right) + w^i v^j\frac{\partial^2 L}{\partial v^i\partial v^j}\right]\mathrm{d}t \\
(\pi_1^2)^*\mathrm{d}L &= \frac{\partial L}{\partial t}\mathrm{d}t + \frac{\partial L}{\partial q^k}\mathrm{d}q^k + \frac{\partial L}{\partial v^k}\mathrm{d}v^k \ ,
\end{aligned}$$

we obtain

$$\begin{aligned}
\imath_1^* d_T\Theta_{\mathcal{L}} - (\pi_1^2)^*\mathrm{d}L &= \left(\frac{\partial^2 L}{\partial v^i\partial v^k}w^i + \frac{\partial^2 L}{\partial q^i\partial v^k}v^i + \frac{\partial^2 L}{\partial t\partial v^k} - \frac{\partial L}{\partial q^k}\right)(\mathrm{d}q^k - v^k\mathrm{d}t) \\
&= \left[\frac{d}{dt}\left(\frac{\partial L}{\partial v^k}\right) - \frac{\partial L}{\partial q^k}\right](\mathrm{d}q^k - v^k\mathrm{d}t) \ .
\end{aligned}$$

Now, suppose that there are external forces operating on the Lagrangian system $(J^1\pi, \mathcal{L})$. A force depending on velocities is a section $F\colon J^1\pi \longrightarrow \mathcal{C}_\pi$. As above, the corresponding Euler-Lagrange equations are a system of second order differential equations on Q, given in implicit form by the submanifold D_F of $J^2\pi$ determined by:

$$D_F = \{\hat{p} \in J^2\pi \mid (\imath_1^* d_T\Theta_{\mathcal{L}} - (\pi_1^2)^*\mathrm{d}L)(\hat{p}) = (F\circ\pi_1^2)(\hat{p})\} = \{\hat{p} \in J^2\pi \mid \mathcal{E}_{\mathcal{L}}(\hat{p}) = (F\circ\pi_1^2)(\hat{p})\} \ .$$

A section $\phi\colon \mathbb{R} \longrightarrow \mathbb{R}\times Q$ is a solution to the Lagrangian system if, and only if,

$$\mathcal{E}_{\mathcal{L}}(j^2\phi) = (\pi_1^2)^*[(F\circ\pi_1^2)(j^2\phi)] = (\pi_1^2)^*F(j^1\phi) \ . \tag{47}$$

In natural coordinates we have

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial v^k}\right) - \frac{\partial L}{\partial q^k}\right](\mathrm{d}q^k - v^k\mathrm{d}t) = F_j(\mathrm{d}q^j - v^j\mathrm{d}t) \ .$$

Acknowledgments

We acknowledge the financial support of *Ministerio de Educación y Ciencia*, Projects MTM2005-04947, MTM2004-7832, and S-0505/ESP/0158 of the CAM. One of us (MBL) also acknowledges the financial support of the FPU grant AP20040096. We thank Mr. Jeff Palmer for his assistance in preparing the English version of the manuscript.

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